SOME APPLICATIONS OF FIXED POINT THEOREMS IN CONTROL THEORY

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CERTIFICATE

Certified that this work, ''Some Applications of Fixed Point Theorems in Control Theory'' by Mr. Subhendu Das, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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SYNOPSIS

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SOME APPLICATIONS OF FIXED POINT THEOREMS IN CONTROL THEORY.

Many control problems can be represented in the form of Integral Equations (IE). Noting that these are special cases of operator equations, a functional analytic view point (mainly the constructive approaches) to study the qualitative properties, like existence and uniqueness of solutions, of these equations, has been adopted. Emphasis has been given to show how these techniques can be applied to control problems rather than giving best possible results or solving complicated large practice oriented problems. Several numerical examples have been included to demonstrate the procedures.

For Boundary Value Problems (BVP), of both two point and multipoint cases, a simple equivalent IE representation has been given. The technique involved can accomodate large class of constraints on the system dependent variable (called state) and other parameters.

Using this technique, as illustration, a very general non-linear controllability problem has been given an equivalent IE representation. With the help of some modifications of Banach's fixed point theorem, the equivalent IE for the BVP has been studied.

The most important aspect of this work is the study of the controllability problems. For this problem, our approach is motivated by the fact that these are essentially two point BVPs, where an unknown parameter, called control, is to be found out, which will steer the system between any given pair of states. A set of IEs has been given, the solution of which represents a solution of the controllability problem. Using fixed point theorems on these IEs existence of atleast one control has been assured. The method presented here also finds out the numerical values of the control. A very simple result has been obtained for the controllability of non-linear system, when the dimensions of both the state and control vectors are identical.

The stability problems have also been investigated by fixed point theorems. For the non linear forced system, when the forcing function is dependent on both state and independent variables (called time)of the system, Banach's

fixed point theorem has been used. And for the open loop systems, that is, when the forcing function is dependent only on time, a modification of Kakutani's theorem to Banach spaces, given by Bhonenblust and Karlin, has been used without any convexity conditions on the system.

TABLE OF CONTENTS

CHAPTER - ONE

MATHEMATICAL PRELIMINARIES

In this chapter we present the necessary mathematical background required for the understanding of the following chapters. The reader wishing to obtain a more detailed of these concepts should refer to [1 - 8]. We will be using the following basic set theoretic notations :

$\Rightarrow$	implies	$\ni$	such that
$\Leftrightarrow$	implies and implied by	$\exists$	there exists
$\in$	belongs to	$\forall$	for all
$\subset$	subset of	iff	if and only if

BANACH SPACES 1.1

Linear Spaces

Let X be a non-empty set. Assume that

(i) there is a rule that will allows us to construct, for every two elements $x, y \in X$ another element $z \in X$, called the sum of the elements x and y and denoted by $z = x + y$.

(ii) there is a rule that will allow us to construct for every element $x \in X$ and every scalar α an element $u \in X$ called the product of the element x and the scalar α denoted by $\alpha x = u$.

DEFINITION - 1.1.1

A set X is called a linear space if additions and scalar multiplications are defined on X and the following rules hold:

(1) $x + y = y + x$ for all $x, y \in X$

(2) $(x+y)+z = x+(y+z)$ for all $x,y,z \in X$

(3) There exists an element $0 \in X$ (the zero element) such that $x + 0 = x$ for any $x \in X$.

(4) For any $x \in X$, $\exists$ $y \in X$ $\ni$ $x + y = 0$.

(5) $1x = x$ for all $x \in X$.

(6) $\alpha(\beta x) = (\alpha\beta)x$ for any $x \in X$ and any α and β.

(7) $(\alpha+\beta)x = \alpha x + \beta x$ for any $x \in X$ and any α and β.

(8) $\alpha(x+y) = \alpha x + \alpha y$ for any $x,y \in X$ and any α.

A linear space is called a real or complex according to whether the scalars are the real or complex number system.

DEFINITION 1.1.2

A linear space is called a normed linear space if a rule exists, which associates with every element $x \in X$ a real number (called the norm of an element x and denoted by $||x||$). This rule must obey the following conditions (norm axioms) :

(1) $||x|| \geq 0$, $||x|| = 0 \Leftrightarrow x = 0$.

(2) $||x+y|| \leq ||x|| + ||y||$.

(3) $||\alpha x|| = |\alpha|(||x||)$

where α is any scalar.

Henceforth we will assume that X is a normed linear space.

Open and Closed Sets:

DEFINITION 1.1.3

If x_0 is a point of X and r a positive real number, then the open sphere $S_r(x_0)$ with centre x_0 and radius r is the subset of X defined by

$$S_r(x_0) = \{ x : ||x - x_0|| < r \}$$

DEFINITION 1.1.4

A subset G of X is called an open set if given any point x in G there exists a positive real mumber r such that $S_r(x) \subset G$ i.e. if each point of G is the centre of some open sphere contained in G.

THEOREM 1.1.5

In any normed space, each open sphere is an open set.

THEOREM 1.1.6

A subset G of X is open $\Longleftrightarrow$ it is a union of open spheres.

THEOREM 1.1.7

Any union of open sets in X is open and any finite intersection of open sets in X is open.

DEFINITION 1.1.8

Let A be a subset of X a point x in X is called a limit point of A if each open sphere centered on x contains atleast one point of A different from x.

DEFINITION 1.1.9

A subset F of X is called a closed set if it contains each of its limit points.

THEOREM 1.1.10

A subset F of X is closed $\Leftrightarrow$ its complement F^c is open.

THEOREM 1.1.11

Any intersection of closed sets and any finite union of closed sets in X are closed sets.

DEFINITION 1.1.12

If x_0 is a point in X, and r a non-negative real number, then the closed sphere $S_r[x_0]$ with centre x_0 and radius r is the subset of X, defined by

$$S_r[x_0] = \{x : ||x - x_0|| \leq r\}$$

THEOREM 1.1.13

In a normed space each closed sphere is a closed set.

Completeness:

Let $x_n = \{x_1, x_2, \ldots, x_n, \ldots\}$ be a sequence of points in X. We say that $\{x_n\}$ is convergent if there exists a point x in X and for

each $\epsilon > 0$ there exists a positive integer n_0 such that

$$n \geq n_0 \Rightarrow ||x_n - x|| \leq \epsilon$$

Every convergent sequence $\{x_n\}$ has the following property: for each $\epsilon > 0$, there exists a positive integer n_0 such that $m,n \geq n_0 \Rightarrow ||x_m - x_n|| \leq \epsilon$. A sequence with this property is called a Cauchy sequence. It is to be noted here that not every Cauchy sequence is convergent because for convergence it is necessary that the limiting element should belong to the set.

DEFINITION 1.1.14

A normed space is complete, if every Cauchy sequence in it is convergent.

THEOREM 1.1.15

Let X be a complete normed space and let Y be a subspace of X. Then Y is complete $\Leftrightarrow$ it is closed.

DEFINITION 1.1.16

A Banach space is a complete normed linear space.

We will use the word 'B-space' and 'Banach space' interchangebly.

OPERATORS: 1.2

In what follows, we assume that X and Y are Banach spaces.

DEFINITION 1.2.1

A function A is called an operator if to each element x in some subset D of X, A assigns one element $y \in Y$. The set D is called the domain of A, while the set of all elements $y = Ax$ for all $x \in D$ is called the range of A.

DEFINITION 1.2.2

An operator A is called bounded in X if there exists a constant M such that $||Ax|| \leq M||x||$ for all $x \in X$.

DEFINITION 1.2.3

An operator A is called continuous if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $||x_1-x_2|| \leq \delta$ implies $||Ax_1 - Ax_2|| \leq \epsilon$ for all $x_1, x_2 \in X$.

DEFINITION 1.2.4

An operator A is called additive if $A(x_1+x_2) = Ax_1+Ax_2$ for every $x_1, x_2 \in X$, and homogeneous if $A(\alpha x) = \alpha Ax$ for every scalar α and every $x \in X$. An operator which is both additive and homogeneous is called linear.

DEFINITION 1.2.5

The norm of a bounded linear operator A is the smallest number M which satisfies $||Ax|| \leq M\,||x||$ for all $x \in X$. It is denoted by $||A||$.

It can be shown that $||A||$ is given by

$$||A|| = \sup_{x \neq 0} \frac{||Ax||}{||x||} = \sup_{||x||=1} ||Ax||$$

where Sup stands for supremum i.e. the least upper bound. Following theorems give some important properties of linear operators in Banach Spaces.

THEOREM 1.2.6

If A is a linear operator from X into Y which is continuous at some $x_0 \in X$, then A is continuous at every point $x \in X$.

THEOREM 1.2.7

Let X,Y be Banach spaces and A a linear operator from X into Y. Then A is bounded $\Leftrightarrow A$ is continuous.

THEOREM 1.2.8

Additive and continuous operators are linear.

By $[X,Y]$ we will denote the class of all linear continuous operators mapping X into Y. $[X,X]$ will be denoted by $[X]$.

THEOREM 1.2.9

Let X and Y be Banach Spaces. Let the norm of $A \in [X,Y]$ be defined as in Definition 1.2.5.

Then the space $[X,Y]$ is a Banach space with respect to the point wise linear operations.

DEFINITION 1.2.10

Let A be an operator mapping X into Y. We say that the operator is an onto mapping if corresponding to any element $y \in Y$ $\exists$ an $x \in X \ni Ax = y$. If under the operation A two different elements in X always have different images then A is called an one-to-one mapping of X into Y.

THEOREM 1.2.11

Let X and Y be Banach spaces and A a linear operator mapping X onto Y. Let a positive number m exist such that for every $x \in X$

$$||Ax|| \geq m \; ||x||$$

Then the operator A has a linear inverse A^{-1}, where

$$||A^{-1}|| \leq \frac{1}{m} .$$

MEASURE AND INTEGRATION 1.3

Following is a brief exposition of the modern theory of integration. We follow the lines of Halmos [5] .

Measurable sets

DEFINITION 1.3.1

Let X be any non empty set. A family S of

subsets of X is said to be a σ-algebra if it satisfies the following conditions

(i) 0(empty set) and $X \in S$

(ii) $A \in S$, $B \in S \Rightarrow A - B \in S$

(iii) if $\{A_n\}$ is an infinite sequence of sets in S then

$$\bigcup_{n=1}^{\infty} A_n \in S$$

We assume S to be the smallest σ- algebra on X.

DEFINITION 1.3.2

A measure is an extended real valued set function μ, defined on a σ-algebra S such that

(i) $\mu(E) \geq 0 \qquad E \in S$

(ii) $\mu(0) = 0$

(iii) $\{A_n\} \in S$, $A_n \cap A_m = 0$ for $m \neq n$. $\qquad \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$

DEFINITION 1.3.3

An outer measure μ^* for any arbitrary set E in S is defined by

$$\mu^*(E) = \text{Inf } \{\mu(F) : F \supset E,\ F \in S\}$$

In a similar way an inner measure μ_* is defined by

$$\mu_*(E) = \text{Sup } \{\mu(F) : F \subset E,\ F \in S\}$$

DEFINITION 1.3.4

A set E in S is said to be measurable if its inner and outer measures coincide with its measure

i.e. if $\mu^{*}(E) = \mu(E) = \mu_{*}(E)$

It can be shown that all the sets in S is measurable.

A statement is said to hold almost everywhere (a.e) if it is true except over a set of measure zero.

DEFINITION 1.3.5

A sequence $\{f_n\}$ of a.e. finite valued measurable functions converges in measure to the measurable function f if for every $\epsilon > 0$

$$\lim_{n \to \infty} \mu(\, x : |f_n(x) - f(x)| \geq \epsilon \,) = 0.$$

DEFINITION 1.3.6

A measurable space is a set X equipped with a σ - algebra S of subsets of X .

Let X be the real line, P, the class of all bounded semiclosed interval of the form $[a,b)$, S the σ - algebra generated by P. Let μ be the set function defined on the sets of S by $\mu([a,b)) = b - a$. Then it can be shown that μ is a measure on S. This measure is known as Lebesgue measure. Obviously this is a particular case of measure, so most of the results hold for this case, without probably any change.

Integrable Function

DEFINITION 1.3.7

A real valued function is measurable if for any real number c the set $\{x: f(x)<c\}$ is measurable.

DEFINITION 1.3.8

A function f defined on a measurable space X, is called simple if there is a finite disjoint class $\{E_1, \ldots E_n\}$ of measurable sets and a finite set $\{\alpha_1, \ldots \alpha_n\}$ of real numbers such that

$$f(x) = \begin{cases} \alpha_i & \text{if } x \in E_i \quad i = 1, 2, \ldots n \\ 0 & \text{if } x \notin E_1 \; E_2 \ldots \; E_n \end{cases}$$

THEOREM 1.3.9

Every extended real valued measurable function f is the limit of a sequence $\{f_n\}$ of simple functions.

DEFINITION 1.3.10

A simple function is integrable if $\mu(E_i) < \infty$ for every index i for which $\alpha_i \neq 0$. The integral of f is defined by

$$\int f \, d\mu = \sum_{i=1}^{n} \alpha_i \, \mu(E_i)$$

DEFINITION 1.3.11

A sequence of integrable simple functions is said to be mean fundamental if

$$\mu(f_n, f_m) \to 0 \quad \text{as} \quad n, m \to \infty$$

where $\mu(f,g) = \int |f-g| \, d\mu$.

DEFINITION 1.3.12

An almost everywhere finite valued measurable function is said to be integrable if there exists a mean fundamental sequence $\{f_n\}$ of integrable simple functions which converges in measure to f. The integral of f is defined by

$$\int f\,d\mu = \lim_{n \to \infty} \int f_n d\mu .$$

THEOREM 1.3.13

A bounded measurable function is integrable.

DIFFERENTIAL EQUATIONS 1.4

In this section we present some standard results on initial value problem of ordinary differential equations which will be helpful in understanding boundary value problems.

Lipschitz Condition

Let f be a function defined over the set $S \subset R^n$, where R^n denotes the real n dimensional space, with values in R^n. We say that f satisfies the Lipschitz (Lip) condition on S if there exists a constant $L \geq 0$ such that

$$||f(x) - f(y)|| \leq L||x - y||$$

for all $x, y \in S$. The constant L is known as Lipschitz constant. A function satisfying Lipschitz condition is

called Lipschitzian. Note that a Lipschitzian function is continuous.

If the function f possesss first partial derivative with respect to all the components of x over its domain of definition, then it can be shown, that the Lipschitz constant L can be taken as

$$L = \max_i \sum_{j=1}^{n} \sup_x |a_{ij}(x)|$$

where a_{ij} s are components of the Jacobian matrix of f.

Standard Results

We assume the following properties for the function f.

(i) $f: I \times D \to R^n$ where I and D are open subsets of R and R^n respectively.

(ii) f is continuous in $x \in D$, for each fixed $t \in I$ and measurable in $t \in I$ for each fixed $x \in D$.

(iii) there exists a Lebesgue integrable function $m(t)$ such that $\|f(t,x)\| \leq m(t)$ for all $x \in D$ and almost everywhere in I.

The conditions (ii) and (iii) are known as Caratheodory conditions.

DEFINITION 1.4.1

Let S be some subset of $I \times D$. Then by the solution of the initial value problem

$$\dot{x} = f(t,x) \qquad x(t_o) = a \tag{1.4.2}$$

we mean an absolutely continuous function $\phi(t)$ such that

(i) $(t,\phi(t)) \in S$,

(ii) $\phi(t_o) = a$

(iii) $\dot{\phi}(t) = f(t, \phi(t))$ for all t , except over a set of Lebesgue measure zero, of the interval for which $\phi(t)$ is defined.

LEMMA 1.4.3

The initial value problem (1.4.2) is equivalent to the following integral equation

$$x(t) = a + \int_{t_o}^{t} f(s,x(s))ds \tag{1.4.4}$$

The above lemma assures that instead of studying the system (1.4.2) we can study the integral equation (1.4.4) which in some cases may be easier to handle with.

THEOREM 1.4.5

Let $f(t,x)$ satisfy all the conditions mentioned above. Then there exists a solution $\phi(t)$, defined over some interval of I, of the initial value problem (1.4.2). In addition if $f(t,x)$ satisfies the Lipschitz condition in x for all $t \in I$, the solution is unique.

LINEAR BOUNDARY VALUE PROBLEM 1.5

The linear boundary value problem for ordinary differential equation has been studied almost exhaustively. Any standard book on differential equation will give good record of these results, see for example [6,7]. The material presented in this section closely follows Falb [8].

Consider the linear two point boundary value problem (TPBVP)

$$\dot{x}(t) = A(t)\ x(t) + f(t) \tag{1.5.1}$$

$$Mx(o) + Nx(1) = a \tag{1.5.2}$$

where $x(t)$, $f(t)$, a are n-vectors, A, M and N are nxn matrices. Assume that $A(t)$ and $f(t)$ are defined and measurable over an open interval I containing the closed interval $[0,1]$, also that there exists an integrable function $m(t)$ on I such that $||A(t)|| \leq m(t)$, $||f(t)|| \leq m(t)$.

DEFINITION 1.5.3

An n-vector valued function $\phi(t)$ is called a solution of the above TPBVP if

(i) $\phi(t)$ is an absolutely continuous function

(ii) $\dot{\phi}(t) = A(t)\ \phi(t) + f(t)$ for almost all t in $[0,1]$

(iii) $M\ \phi(o) + N\phi(1) = a$.

Let $\Phi(t,t_o)$ be the fundamental matrix of the linear system $\dot{x} = A(t)\,x$ with $\Phi(t_o,t_o) = I$. Let us assume for convinience that $t_o = o$ and $t \in [o,1] \subset I$. Then we know that

$$x(t) = \Phi(t,o)\,x(o) + \int_o^t \Phi(t,\tau)f(\tau)d\tau \qquad (1.5.4)$$

is the solution of the differential equation (1.5.1) for some given initial condition $x(o)$. If this solution is to satisfy the Boundary condition (1.5.2) then we must have

$$Mx(o) + N[\Phi(1,o)x(o) + \int_o^1 \Phi(1,\tau)\,f(\tau)d\tau] = a$$

or $[M+N\ \Phi(1,0)]\ x(o) = a - N\int_o^1 \Phi(1,\tau)f(\tau)d\tau = b(\text{say})$.

Thus we have the following lemmas.

LEMMA 1.5.5

Let S denote the set of all solutions of the TPBVP mentioned above. Then S is non empty iff

$$a - N\int_o^1 \Phi(1,\tau)\ f(\tau)d\tau = b$$

is an element of the range of the operator $[M+N\,\Phi(1,o)]$.

LEMMA 1.5.6

If the matrix $[M+N\ \Phi(1,o)]$ is non singular then the TPBVP defined above has an unique solution which can be expressed in the following form

$$x(t) = H(t)\, a + \int_0^1 G(t, s)\, f(s)\, ds \qquad (1.5.7)$$

where the Green's Matrices H(t) and G(t,s) are given by

$$H(t) = \Phi(t,o)\,[M + N\,\Phi(1,o)]^{-1} \qquad (1.5.8)$$

$$G(t,s) = \begin{cases} H(t)\, M\, \Phi(o,s) & o \leq s \leq t \\ -H(t)\, N\, \Phi(1,s) & t \leq s \leq 1 \end{cases} \qquad (1.5.9)$$

respectively.

CHAPTER - TWO

FIXED POINT THEOREMS

INTRODUCTION 2.1

An operator A mapping a Banach space X into itself is said to have a fixed point if there exists an element $x^* \in X$ such that

$$Ax^* = x^*$$

i.e. the element x^* remains invariant under the operation A. The study of the fixed point of an operator is equivalent to the study of the solution of the operator equation

$$Ax = x \tag{2.1.1}$$

or equivalently

$$Bx = 0 \tag{2.1.2}$$

where $B = A - I$ and 0 is the null element of the B-space X.

There are various theorems, known as fixed point theorems, available for the study of the existence and uniqueness properties of the fixed points of an operator. Most commonly used results of them are the Schauder's [9] and Banach's [9] fixed point theorems.

Newton's method [1] or the Bellman's quasilinearization technique [10] has also been widely used [11] for solving the operator equation (2.1.2). It has been shown that this method is a special case of Kantorovitch's fixed point theorem [1].

In Schauder's fixed point theorem one needs to find a convex closed compact set which will be mapped into itself by the operator. This free choice of convex set makes the theorem very flexible. But in most cases it becomes almost impossible to find out a suitable convex set, specially if the operator is not of general standard form. Moreover this theorem does not give any constructive procedure to compute the fixed point, which is quite often required in applied analysis.

On the other hand Banach's theorem, popularly known as Contraction Mapping Principle (CMP) is very simple to apply and it gives a constructive procedure to find out the fixed point. The method generates a sequence of elements which converges to the solution. The theorem requires that the operator should be Lipschitzian [see section 1.4] with Lipschitz constant less than one. It has been found that many operators exhibit solutions even though their Lipschitz

constants are not less than one. This means that the Banach's theorem usually puts a very strong condition on the operator.

To avoid this difficulty many modifications have been presented. In this chapter we present some of these results, which are helpful in solving problems. A very interesting result in this connection is the one due to L.M. Grave [12] . We show here that the modified contraction mapping principle as given by Falb [8] is nothing but a special case of the Grave's theorem. We will be using these results for the analysis of Boundary value, Controllability and Stability problems for the non linear ordinary differential equation in the following chapters.

The other widely used method, the Newton's method, has a faster rate of convergence of the sequence it generates, than that of CMP, but at the expense of large number of computations, specially in the evaluation of the derivatives, an oftenly unwanted computation. Otherwise this method does not have any special advantage over the CMP. Indeed the problems that can be solved by CMP can not always be solved by Newton's method, on the other hand the converse is true. A detailed and comparative study of these methods can be found in [8].

BANACH'S THEOREM 2.2

Throughout in this chapter we assume that X is a Banach space (B-space) with the norm represented by $\|\cdot\|$. By small letters x, y, arbitrary general elements of X will be represented. Capital letters A and B will denote operators defined over X with values in X.

THEOREM 2.2.1
(Banach)

Let $A: X \to X$ be an operator satisfying the conditions

(i) $AS \subset S$ where S is a closed subset of X

(ii) $\|Ax - Ay\| \leq \alpha \|x - y\|$ for all $x, y \in S$ and with $0 \leq \alpha < 1$

Then the operator A has a unique fixed point in S i.e. there exists a unique element $x^* \in S$ such that

$$Ax^* = x^*$$

And based on any $x_0 \in S$ the sequence generated by

$$x_{n+1} = Ax_n$$

converges to x^* with respect to the normed topology of X.

In many cases, because of the method used, it turns out that the condition (i) of the above theorem

puts a stronger restriction on α than what is mentioned in condition (ii) specially when the operator is not defined over the whole space.

Suppose $S = S_r$ a closed sphere of radius r with centre on the origin, and let A be defined over S_r with $A0 = a$. Then for any $x \in S$ condition (i) requires that $Ax \in S$. We have

$$||Ax|| = ||Ax - A0 + a|| \leq ||Ax - A0|| + ||a||$$

using condition (ii) in the first term of the right hand side of the above expression, we get

$$||Ax|| \leq \alpha||x - 0|| + ||a|| \leq \alpha r + ||a||$$

Thus for Ax to belong to S_r we require

$$\alpha r + ||a|| \leq r$$

$$\text{i.e.} \quad \alpha \leq 1 - \frac{||a||}{r}$$

The above shows that α should be usually less than what is demanded by condition (ii). In view of the above fact in most of the cases it is sufficient to verify the condition (i).

In future we will refer the above theorem as Banach's fixed point theorem or the contraction mapping

principle. The condition (i) is known as self mapping property and the (ii) as contraction property. The condition (ii) is also known as Lipschitz condition with α as Lipschitz constant.

MODIFICATIONS 2.3

As mentioned before the restriction on α in CMP is quite strong. Following theorem due to Chu and Diaz [13] and Kantorovitch [1] gives a modification over the above result.

THEOREM 2.3.1
(Chu, Diaz, Kantorovitch)

Let A and B be two operators defined over a non-empty set S with values in S. Let B possess a right inverse B^{-1}. Then the operator A has a fixed point iff $B^{-1}AB$ has a fixed point.

The essential idea behind the above said theorem is that if A is not a contraction operator and if there exists a B operator such that $B^{-1}AB$ is contraction, then the fixed point of A can be computed using the CMP over the composite operator $B^{-1}AB$. It is to be noted that if x^* is a fixed point of $B^{-1}AB$ then Bx^* is the fixed point of A. The above follows from

$$x^* = B^{-1}ABx^* \Longrightarrow ABx^* = Bx^*$$

In some cases it may be possible to reduce the Lipschitz constant of the operator A by suitably redefining the norm of the underlying space, so that the operator A becomes a contraction with respect to the new norm. It has been shown in [13] that this method is nothing but a special case of the theorem 2.3.1.

Following theorem can also be considered as a modification over CMP. A very interesting applica - tion of this result can be found in [14] .

THEOREM ([14]) 2.3.2

The fixed point of the operator A is the fixed point of the operator A^n and vice versa, for any positive integer $n \geq 1$.

Thus if for some integer n the operator A^n is contraction, then using CMP on A^n, the fixed point of A can be numerically computed.

GRAVE'S THEOREM 2.4

The theorem due to L.M. Grave [12] may be considered as a significant modification over the Banach's theorem. Many problems which cannot be solved directly by using Banach's theorem can be solved by

Grave's theorem. This theorem is based on the following lemma due to Banach [15].

LEMMA 2.4.1
(Banach)

If B is a continuous linear operator of the space X onto the whole space Y, there exists a number $\beta > 0$ such that for each $y \in Y$ there is an $x \in X$ with $y = Bx$ and

$$||x|| \leq \beta \; ||y||$$

THEOREM 2.4.2
(Graves)

Let A be a continuous operator defined for $||x|| \leq r$ with values in Y and $A0 = 0$. Let B be another linear continuous transformation of X onto the whole space Y such that

$$||Ax_1 - Ax_2 - B(x_1 - x_2)|| \leq \alpha \; ||x_1 - x_2||$$

whenever x_1 and x_2 belong to S_r, where $1/\alpha$ is greater than the constant β of lemma 2.4.1. Then the equation $y = Ax$ has a solution x in S_r whenever

$$||y|| \leq r(1 - \alpha\beta)/\beta \; .$$

It has been shown [12] in the proof that the sequence $\{x_n\}$ generated by the alternate application

of the formulas

$$y_{n-1} = B(x_n - x_{n-1}) \tag{2.4.3}$$

$$y_n = B(x_n - x_{n-1}) - Ax_n + Ax_{n-1} \tag{2.4.4}$$

where $x_o = 0$ and $y_o = y$, converges to the solution of the equation

$$y = Ax \quad .$$

The difficulty with the above theorem is to find out the value of x_n, when y_{n-1} and x_{n-1} are known, by using equation (2.4.3). But if B is invertible this will be greatly simplified. In this case the value for β of Banach's lemma can be taken as

$$\beta = || B^{-1} ||$$

If B is invertible the Grave's theorem can be rewritten in the following form. For convenience let us use the notation

$$A \in (\mathrm{Lip}, S, \alpha) \Leftrightarrow ||Ax - Ay|| \leq \alpha \, ||x - y||$$

for all $x, y \in S$.

<u>THEOREM</u> 2.4.5

Let X be a Banach space and S_r a closed subset of it. Let $A : S_r \to X$ and let B be a linear

invertible operator defined over X with values in X. Let $||B^{-1}|| \leq \beta$. Assume that the following conditions hold

(i) $[A + B] \in (\text{Lip}, S_r, \alpha)$

(ii) $[A + B]\, S_r \subset S_{r/\beta}$

(iii) $\alpha\beta < 1$.

Then there exists a unique element $x^* \in S_r$ such that

$$Ax^* = 0$$

and based on any $x_0 \in S_r$, the sequence generated by

$$y_n = [A + B]\, x_n \qquad (2.4.6)$$

$$y_n = B\, x_{n+1}$$

converges to x^*.

<u>Proof</u> (Sketch):

Let $x_0, x_1, \ldots x_n$ belong to S_r. Then

$$||x_{n+1}|| = ||B^{-1}y_n|| \leq \beta||y_n|| = \beta||[A+B]\, x_n||$$

$$\leq \beta \cdot \frac{r}{\beta} = r \Rightarrow x_{n+1} \in S_r$$

$$||x_{n+1} - x_n|| = ||B^{-1}y_n - B^{-1}y_{n-1}|| \leq \beta||y_n - y_{n-1}||$$

$$= \beta||[A+B]\, x_n - [A + B]\, x_{n-1}|| \leq \alpha\beta||x_n - x_{n-1}||$$

Since $\alpha\beta < 1$,

the above implies $\{x_n\}$ is Cauchy. Let $x_n \to x^*$, obviously $x^* \in S_r$ because S_r is a closed set. Define $y^* = [A+B]\, x^*$. Because of the continuity of $[A+B]$, which follows from condition (i),

$$\lim y_n = \lim\, [A+B]x_n = [A+B] \lim x_n = [A+B]x^* = y^*$$

and hence from (2.4.6) and (2.4.7) we have

$$\left.\begin{aligned} y^* &= [A+B]\ x^* \\ y^* &= B\, x^* \end{aligned}\right\} \Longrightarrow A\, x^* = 0$$

Q.E.D.

It is easy to see that the sequence $\{x_n\}$ generated by (2.4.6) and (2.4.7) can be generated in a simpler way by

$$x_{n+1} = B^{-1}[\,A + B\,]\, x_n \tag{2.4.8}$$

Thus we have the following theorem which Falb calls as modified contraction mapping principle [8] and is also available in [16].

THEOREM 2.4.9
(Falb, Krasanoselskii)

Let A and B be maps of X into X. Let B be invertible. Then the fixed point of $B^{-1}[A + B]$ is the solution of

$$Ax = 0\ .$$

Thus if B is linear, the theorem 2.4.9 can be considered as a special of Grave's theorem. In the following chapters we will use Graves theorem in the form given by theorem 2.4.5 or 2.4.9, depending on the convenience. We observe that these two theorems are identical when B is linear.

EXAMPLE 2.5

We illustrate the application of the above results by an example from integral equations. Consider the space C, the class of all continuous functions defined over the real interval $[0,1]$ with values in X, an arbitrary Banach space with norm $||\cdot||_X$. Let the norm in C be defined by

$$||x||_c = \sup_{t \in [0,1]} \{ ||x(t)||_X : x(t) \in X, x \in C \}$$

It can be shown that with the above norm the class C is a Banach space.

Let the function $F(t,x): [0,1] \times S_r \to X$, $S_r \subset X$ satisfy the following conditions

(i) $F(t,x)$ is continuous in $x \in S_r$ for each fixed $t \in [0,1]$ and measurable in $t \in [0,1]$ for each fixed $x \in S_r$.

(ii) There exists an integrable function $m(t)$ such that $||F(t,x)||_X \leq m(t)$ for all $x \in S_r$.

(iii) $F(t,x)$ satisfies the following uniform Lipschitz condition

$$||F(t,x)- F(t,y)||_X \leq L||x-y||_X \text{ for all } x,y \in S_r$$

and all $t \in [0,1]$.

(iv) $F(t,0) = 0$ for all $t \in [0,1]$.

Let $q(t)$ be a continuous function belonging to the class C with the norm

$$|| q ||_C \leq a$$

where a is any non-negative real number.

We are interested in studying the existence and uniqueness properties of the solutions of the following integral equation

$$x(t) = q(t) - x(1) + \int_0^t F(s,x(s))\, ds \qquad t \in [0,1]$$

By solution we mean a function $\phi \in C$ which satisfies the above equation. In above $x(1)$ is not known. Clearly because of the presence of $x(1)$ direct application of CMP will fail, however small the Lipschitz constant (for F) may be . Formally we have the following result.

Assume that the function $F(t,x)$ and $q(t)$ satisfy the conditions mentioned above. Let $o \leq L \leq \frac{2}{3} - \frac{a}{r}$. Then the above equation has a unique solution.

To verify this we proceed in the following way, along the lines of the theorem 2.4.5. Define the operator A by

$$y = Ax \qquad y(t) = q(t) - x(t) - x(1) + \int_0^t F(s,x(s))ds$$

The obvious choice for B is

$$y = Bx \qquad y(t) = x(t) + x(1)$$

and hence $[A + B]$ takes the form

$$y = [A + B]\, x \qquad y(t) = q(t) + \int_0^t F(s,x(s))ds$$

Clearly B is linear and invertible, B^{-1} is given by

$$x = B^{-1}y \qquad x(t) = y(t) - \frac{1}{2}\, y\,(1)$$

which gives

$$||x||_c \leq ||y||_c + \frac{1}{2}\, ||y||_c = \frac{3}{2}\, ||y||_c$$

thus

$$\beta = 3/2\ .$$

For any $x,y \in C$, $y(t),x(t) \in S_r$, we have

$$||[A + B]x\ (t) - [A+B]\ y(t)||_X$$

$$= ||\int_0^t [F(s,x(s)) - F(s,y(s))]\,ds\,||_X$$

$$\leq \int_0^t ||F(s,x(s)) - F(s,y(s))||_X ds$$

$$\leq L \int_0^t ||x(s) - y(s)||_X\ ds$$

taking supremum on both the sides, we get

$$||[A+B]x - [A+B]y\ ||_C \leq L\ ||x - y||_C$$

which gives $\alpha = L$. Hence

$$\alpha\beta = L.\ \frac{3}{2} = (\ \frac{2}{3} - \frac{a}{r})\ \frac{3}{2} = 1 - \frac{3}{2}\ .\ \frac{a}{r}\ < 1\ .$$

Lastly

$$||[A+B]x(t)||_X \leq ||q(t)||_X + \int_0^t ||F(s,x(s))||_X\ ds$$

$$\leq a + \int_0^t L||x(s)||_X\ ds$$

and so

$$||[A+B]x||_C \leq a + L||x||_C \leq a + Lr \leq a +(\ \frac{2}{3} - \frac{a}{r})r = \frac{2}{3}r$$

Thus $[A+B]\ x(t) \in S_{r/\beta}$ whenever $x(t) \in S_r$.

And hence by theorem 2.4.5 the given integral equation has a unique solution $x \in C$ with $x(t) \in S_r$.

CHAPTER - THREE

BOUNDARY VALUE PROBLEM

INTRODUCTION 3.1

In control theory, to find out the value of the optimal control, by the help of necessary conditions of maximum principle or calculus of variations, one needs to solve some non-linear Boundary Value Problems (BVP). Also there are some problems which can be directly converted into certain form of BVP. Many other problems of natural science and engineering quite often lead to BVP. So in a natural way this branch of the theory of Differential Equation (DE) has become a very interesting field of research for both the mathematicians and the applied scientists.

There are various methods available in the literature for the study of the qualitative properties of BVP. The methods [17,18,19] which are non-constructive type, mostly deal with certain class of DE with certain definite form of Boundary Conditions (BC). They are based on certain properties of corresponding initial value problems, some differential inequalities or some assumptions on the uniqueness properties of certain BVP. There are few interesting literature [20,21] on the

non solvability of the BVP, these again give results only for certain class of DE. In the next section with the help of an example [22] we will show the typical behaviour the non-linear BVP exhibits with regard to its solution.

Among the various constructive approaches for the study of the properties of the BVP, Quasilineari. zation Technique [10] is very popular. In this method the non-linear BVP is replaced by a sequence of linear problems whose solutions converge to the solution of the original problem. But as mentioned in the last chapter this method does not have any special advantage over the methods discussed there, except in the convergence rate, but at the expense of higher number of computations required. However we are interested in the application of fixed point theorems presented in the last chapter for the study of the existence and uniqueness properties of the solutions of the non-linear boundary value problem (BVP).

Somehow there are very few literature available in this direction. The book [8] gives a very good computational approach for the study of the most general form of the BVP with the help of fixed

point and other functional analytic theorems. In [22] a certain class of DE with certain type of boundary conditions (BC) have been studied by CMP.

The main difficulty in the study of the BVP via fixed point theorems is to give a suitable integral equation (IE) representation of the problem. It seems that the trend in the existing literature [4,7,8,22] is to give the representation with the help of Green's function of some linear BVP. In this chapter we present a very simple IE representation of the non-linear BVP without using the concepts of Green's function. The technique involved in the representation is capable of dealing with wide class of constraints on the dependent variable of the DE. Using the theorems 2.4.5 and 2.4.9 the IEs have been studied. Few numerical examples have been presented to demonstrate the procedure.

BOUNDARY VALUE PROBLEM 3.2

In this section we define the BVP with its various classes and mention some notations and definitions. We also give an interesting example to show the typical behaviour exhibited by the non-linear BVP. Throughout this work we shall be concerned with

ordinary DE. Moreover the independent variable will always be real. It will usually be represented by t and called time. The dependent variable which will usually be represented by x will also always be real and called state. Unless otherwise mentioned explicitely any function defined over the real line with values in some set will be represented by x, whereas $x(t)$ will represent its value at time t.

Types of BVP.

In theory we can define various kinds of BVP, but in this chapter we will be interested only in two-point and multipoint cases.

DEFINITION 3.2.1

The two point boundary value problem consists of the Differential equation

$$\frac{dx}{dt} = \dot{x}(t) = f(t, x(t)) \tag{3.2.2}$$

with the Boundary condition

$$g(x(a), x(b)) = 0 \tag{3.2.3}$$

where f and g are suitable vector valued functions and $t \in [a,b] \subset R$.

EXAMPLE 3.2.4

Following are the two very important special cases of the two point BVP defined above

$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) \qquad (3.2.2)$$

with the boundary conditions

$$x(a) = A \qquad x(b) = B$$

or

$$x(a) = A \qquad \dot{x}(b) = B$$

The above two problems have been studied extensively in many literature [22] .

DEFINITION 3.2.5

The multipoint boundary value problem consists of the DE(3.2.2) and the boundary condition given by

$$h(x(a_1), x(a_2), \ldots . x(a_n)) = 0 \qquad (3.2.6)$$

where h is some suitable vector function and $a = a_1 < a_2 \ldots < a_{n-1} < a_n = b \quad t \in [a,b] \subset R$.

EXAMPLE 3.2.7

As an example we may have the following third order differential equation

$$\dot{x}_i(t) = f_i(t, x_1(t),\ x_2(t),\ x_3(t)) \qquad i = 1,2,3$$

with the boundary condition given by

$$x_1(a_1) = A_1 \quad x_2(a_2) = A_2 \quad x_3(a_3) = A_3$$

where $a = a_1 < a_2 < a_3 = b$ and $t \in [a,b]$. The above BC may take various combinations of x_i and a_i. For example, we may have

$$x_1(a_1) = A_1 \qquad x_2(a_3) = A_2 \qquad x_3(a_2) = A_3$$

DEFINITION 3.2.8

By solution of the BVP we will mean an absolutely continuous function $\phi(t)$ which satisfies the DE (3.2.2) for almost all $t \in [a,b]$ and the BC (3.2.3) or (3.2.6) according to whether the problem is two point or the multipoint.

Throughout in this chapter, without any loss of generality, we will assume that $[a,b] = [0,1] \subset R$ and that $0 = a_1 < a_2 \ldots < a_n = 1$.

Notations

R^n will always represent the real space of dimension n, R^1 will be denoted by R. The symbol C^n will denote the class of all continuous

functions defined over $[o,1] \subset R$ with values in R^n. C^1 will be denoted by C. The class of functions in C^n which takes values in $S \subset R^n$ will be denoted by $C^n[S]$ i.e.

$$C^n[S] = (\emptyset : \emptyset \in C^n , \emptyset(t) \in S, S \subset R^n)$$

$||\cdot||_n$ will represent the norm in R^n, and $|\cdot|$ will represent $||\cdot||_1$. We define

$$||x(t)||_n = \max_i \{|x_i(t)|: i = 1,2,\ldots n\}$$

where $x_i(t)$ are the components of n-dimensional vector $x(t)$. The norm in C^n, denoted by $||\cdot||_c$, is defined as

$$||\emptyset||_c = \sup_t ||\emptyset(t)||_n \quad t \in [o,1]$$

$$= \max_i \{\sup_t |\emptyset_i(t)|: t \in [o,1] \quad i = 1,2,\ldots n\}$$

where $\emptyset_i(t)$ is the ith component of $\emptyset(t)$.

The norm on the finite dimensional operator $A(t)$ in R^n, a matrix of size n x n, with elements $a_{ij}(t) \in C$ is sdefined by

$$||A||_c = \max_i \{\sum_{j=1}^{n} \sup_t |a_{ij}(t)|: t \in [o,1] , i = 1,2,\ldots n\}$$

In a similar way the norm on $G(t,s)$, a matrix of order n x n, with elements $g_{ij}(t,s)$ each defined for $t,s \in [o,1]$

with values in R is given by

$$||G||_c = \max_i \{ \sum_{j=1}^{n} \operatorname{Sup}_{t,s} |g_{ij}(t,s)| ; t,s \in [o,1] \quad i=1,2,\ldots n \}$$

If there is no comfusion possible all the norms except the one in R will be represented by $||\cdot||$.

The symbol S_r will stand for a closed sphere of radius r centred on origin, i.e.

$$S_r = \{x : ||x||_n \leq r, \; x \in R^n \}$$

DEFINITION 3.2.9

Let I and D be open subsets of R and R^n respectively and let $[o,1] \subset I$. The function $f(t,x)$ will be said to satisfy the property (H) if all of the following conditions are satisfied.

(i) $f(t,x) : I \times D \to R^n$ is continuous in $x \in D$ for each fixed $t \in I$ and measurable in t for each fixed $x \in D$.

(ii) There exists an integrable function $m(t)$ such that

$$||f(t,x)||_n \leq m(t) \quad \text{a.e. in } I$$

and for all $x \in D$.

(iii) f satisfies the uniform Lipschitz condition

$$||f(t,x) - f(t,y)||_n \le L_f||x - y||_n$$

for all $x,y \in D$ and a.e. in I.

(iv) $f(t,0) = 0$ for all $t \in I$.

The first two conditions are known as Caratheodory conditions and the fourth condition will define a free or unforced system.

AN EXAMPLE 3.2.10

In this section we present a study on the solution properties of the following BVP [22]

$$\ddot{y} + |y(t)| = 0 \qquad (3.2.11)$$

$$y(o) = o \qquad y(b) = B \qquad (3.2.12)$$

We first observe the following few properties. A solution of (3.2.11) is a solution of

$$\ddot{y} - y = o \quad \text{if} \quad y(t) \le o \qquad (3.2.13)$$

and

$$\ddot{y} + y = o \quad \text{if} \quad y(t) \ge o \qquad (3.2.14)$$

The solution of (3.2.13) is $y(t) = -a \sinh t$ and of (3.2.14) is $y(t) = a \sin t$. for some arbitrary constant a. Now if at $t = t_o$ $y(t_o) = o$ and $\dot{y}(t_o) < o$, then $y(t) < o$ immediately to the right of t_o and hence is a solution of (3.2.13), and will remain as solution of (3.2.13) for all $t > t_o$, since $-a \sinh t$ is always negative.

In a similar way if at $t = t_o$ $\dot{y}(t_o) > o$ then $y(t) > o$ on some interval to the right of t_o and will thus be the solution of (3.2.14). This will remain as solution upto $t = \pi$, at which $y(\pi) = o$ with $\dot{y}(\pi) < o$ and from then onwards it will follow the equation (3.2.13) in a manner mentioned above.

With this knowledge on the solution of (3.2.11) we can now analyse the BVP.

Case I : $b < \pi$

In this region we have seen equation (3.2.11) has two solutions having both positive and negative values. It is to be noted that zero is the trivial solution of (3.2.13) and (3.2.14). So in this case we have

$$y(t) = \begin{cases} a \sin t & \text{if } B > o \\ -a \sinh t & \text{if } B < o \\ o & \text{if } B = o \end{cases}$$

and hence the solution is unique.

Case II: $b = \pi$.

If $B = o$ there are many solutions given by

$$y(t) = \begin{cases} a \sin t & \text{for all } a > o \\ o & \text{(trivial case).} \end{cases}$$

But if $B > o$ there are no solutions. Since solution of (3.2.13) always remains negative and of (3.2.14)

attains zero at $t = \pi$. But if $B < o$ there exists only one solution $y(t) = -a \sinh t$ where a is to be selected to satisfy $y(b) = B$.

Case III: $b > \pi$.

As above if $B > o$ there are no solutions. If $B = o$ trivial solution is the only solution, because there are no nontrivial solutions which attain zero after $t > \pi$. If $B < o$, then there will be two solutions. One solution is apparent, given by $y_1(t) = -a \sinh t$ with a constant a such that $y_1(b) = B$.

The other solution will be the one which satisfied equation (3.2.14) for $o \leq t \leq \pi$ and then (3.2.13) for $\pi \leq t \leq b$, i.e. the one which remained positive in $o \leq t \leq \pi$, zero at $t = \pi$, and then negative for $t > \pi$. This can be represented in the form

$$y_2(t) = \begin{cases} c_1 \sin t & o \leq t \leq \pi \\ c_2(\sinh t - \tan \pi \cosh t) & \pi \leq t \leq b. \end{cases}$$

the constants c_1 and c_2 should be selected in such a way that $y_2(b) = B$ and $\dot{y}(t)$ is continuous at $t = \pi$.

The above simple example shows, how complicated the behaviour of the solution of a BVP may be. It also shows that not only due to the large value of the Lipschitz constant but also because of the length of the interval and the BC the BVP may fail to exhibit solution.

The application of CMP to the above problem will give the results of Case I. This is because the Lipschitz condition carries very little information of the function. This encouraged some people to work on methods not based on Lipschitz condition[22]. But so far it has not been possible, with the help of these methods, to give very general results. On the otherhand CMP can be quite satisfactorily used to study the very general problem. Modifications of norm may also help to give better results. But it appears that one will be able to give best possible results only for particular cases and not for the general problem.

REPRESENTATION PROBLEM 3.3

We now give the integral equation representation of the two point BVP defined by (3.2.2) and (3.2.3).

THEOREM 3.3.1

Let the function $f(t,x)$ satisfy the first two conditions of the property (H). Let $g(x,y)$ be continuous function mapping $D \times D$ into R^n. Then the Boundary Value Problem

$$\dot{x}(t) = f(t,x(t)) \quad g(x(o), x(1)) = o \tag{3.3.2}$$

has the equivalent representation

$$x(t) = x(o) + g(x(o),\ x(1)) + \int_o^t f(s,x(s))ds \qquad (3.3.3)$$

for almost all $t \in [o,1]$.

Proof: We are to show that any solution of (3.3.2) satisfies (3.3.3). and vice-versa. Let $\emptyset$ be a solution of (3.3.2), then we have

$$\dot{\emptyset}(t) = f(t,\ \emptyset(t)) \qquad g(\emptyset(o),\ \emptyset(1)) = 0$$

which can be written as

$$\emptyset(t) = \emptyset(o) + \int_o^t f(s,\ \emptyset(s))\ ds$$

to the right hand side of the above we can add $g(\emptyset(o),\emptyset(1))$ which is identically equal to zero, without changing any thing, thus

$$\emptyset(t) = \emptyset(o) + g(\emptyset(o),\ \emptyset(1)) + \int_o^t f(s,\emptyset(s))ds$$

And hence $\emptyset$ satisfies (3.3.3).

Conversely, let θ be a solution of (3.3.3) Then

$$\theta(t) = \theta(o) + g(\theta(o),\ \theta(1)) + \int_o^t f(s,\ \theta(s))ds$$

from above it is clear that θ is an absolutely continuous function and therefore differentiable almost everywhere. On differentiation we get

$$\dot{\theta}(t) = f(t,\ \theta(t))$$

also putting $t = o$, the last IE gives

$$\theta(\dot{o}) = \theta(o) + g(\theta(o),\ \theta(1))$$

i.e. $$g(\theta(\bullet),\ \theta(1)) = 0 .$$

Thus θ satisfies equation (3.3.2). Q.E.D.

COROLLARY 3.3.4

Let the hypothesis regarding $f(t,x)$ of the theorem (3.3.1) be satisfied. Then the multipoint boundary value problem

$$\dot{x}(t) = f(t,x(t) \quad g(x(a_1),x(a_2),\ldots x(a_n)) = 0 \quad (3.3.5)$$

where $o = a_1 < a_2 \ldots < a_n = 1$ and g is a suitable vector valued function, has the equivalent representation

$$x(t) = x(o) + \int_o^t f(s,x(s))ds + g(x(a_1),\ldots x(a_n)) \quad (3.3.6)$$

The proof of the above corollary can be given in the same way as has been given for the theorem (3.3.1). Following are few examples of special cases.

EXAMPLE 3.3.7

Consider the BVP in R^2

$$\dot{x}_1(t) = f_1(t,x_1(t),\ x_2(t))$$

$$\dot{x}_2(t) = f_2(t,x_1(t),\ x_2(t))$$

$$x_1(o) = a \quad x_2(1) = b$$

We can form various functions $g(x(o), x(1))$ representing the boundary conditions mentioned above. However a straight forward application gives the following IE representation

$$x_1(t) = a + \int_o^t f_1(s,x_1(s), x_2(s))ds$$

$$x_2(t) = x_2(o) + b - x_2(1) + \int_o^t f_2(s,x_1(s),x_2(s))ds$$

EXAMPLE 3.3.8

Another important class consists of the DEs of example 3.3.7 with the BCs given by

$$x_1(o) = a \quad x_1(1) = b .$$

Again we can construct various g functions, but a direct application gives

$$x_1(t) = a+\int_o^t f_1(s,x_1(s), x_2(s))\ ds$$

$$x_2(t) = x_2(o) + b- x_1(1) + \int_o^t f_2(s,x_1(s),x_2(s))ds$$

REMARKS 3.4

The advantage of the IE representation given by theorem 3.3.1 is that it can include many types of conditions on the system states and parameters present in the system.

For example in the case of periodic solutions the g function will have the form

$$g(x(o),\ x(1)) = x(o) - x(1)$$

If it is required that the integral of the norm of the solution should have some value c then we can take g as

$$g(x) = \int_o^1 ||x(s)||_n \, ds - c$$

The technique of the theorem 3.3.1 can be extended to the following very important class of problems known as controllability problem.

DEFINITION 3.4.1

Let I, D, U be open subsets of R, R^n, and R^m respectively with $[o,1] \subset I$. The function $f(t,x,u)$ will be said to satisfy the property (P) if

(i) $f: I \times D \times U \to R^n$ is continuous in $(x,u) \in D \times U$ for each fixed $t \in I$ and is measurable in $t \in I$ for each fixed $(x,u) \in D \times U$.

(ii) there exists an integrable function $m(t)$ such that

$$||f(t,\ x(t),\ u(t))||_n \leq m\ (t)$$

If there exists a function $u(x^o,x^1) = u \in C^m$, called control, such that by using this u the solution of the DE

$$\dot{x}(t) = f(t,x(t),\ u(t))$$

can be steered from any given x^o to any given x^1 in some finite time interval, then we say that the DE is controllable. More specifically we use the following definition.

DEFINITION 3.4.2

The controllability problem consists of finding out an $u \in C^m[U]$ such that the BVP

$$\dot{x}(t) = f(t,x(t), u(t)) \qquad (3.4.3)$$

$$x(o) = x^o \qquad x(1) = x^1 \qquad (3.4.4)$$

admits a solution for some given x^o and x^1 in D.

Let $\bar{g} : D \times D \to R^n$ be a continuous function such that

$$\bar{g}\ (x,y) = 0 \Longleftrightarrow x = 0,\ y = 0$$

then the BC (3.4.4) is equivalent to

$$\bar{g}\ (x(o) - x^o,\ x(1) - x^1) = 0$$

Hence along the lines of theorem 3.3.1 we have

THEOREM 3.4.5

Let $f(t,x,u)$ satisfy the property (P). Then the Controllability Problem defined by (3.4.3) and (3.4.4) has the equivalent representation

$$x(t) = x(o) + \bar{g}(x(o)-x^o,x(1)-x^1) + \int_o^t f(s,x(s),u(s))ds$$

The proof of the above theorem is very similar to the one given for theorem 3.3.1. We avoid repeatation.

SOLUTION PROBLEM 3.5

Having given the IE representation we can now use the fixed point theorems presented in the second chapter to find out the solution of the problem. We illustrate the solution problem using theorem 2.4.5 when the operator B is taken as linear. This will simplify the presentation. It is to be remembered that B need not be linear. An example with a non-linear B operator has been included to demonstrate the advantage of this type of representation.

As the BVP (3.3.2) is equivalent to the problem (3.3.3), it is sufficient to study the IE. Define the operator A by

$$y = Ax \quad y(t) = x(o) - x(t) + g(x(o)),x(1)) + \int_{o}^{t} f(s,x(s))ds$$

where $f(t,x)$ and $g(x,y)$ satisfy the hypothesis of the theorem 3.3.1. Thus the operator A maps $C^n[D]$ into C^n. We are interested to find out a $\emptyset \in C^n[D]$ such that

$$A\emptyset = 0$$

which will then mean that

$$0 = \phi(o) - \phi(t) + g(\phi(o), \phi(1)) + \int_0^t f(s,\phi(s))\, ds$$

i.e. ϕ will be the solution of the IE (3.3.3).

Let us assume a B operator of the following form

$$y = Bx \qquad y(t) = x(t) - x(o) - Mx(o) - Nx(1) - \int_0^t V(s)x(s)ds$$

where M,N and V(t) are n x n matrices. It is assumed that V(t) is measurable a.e. on I and with the norm dominated by an integrable function over I.

With the above choice of B the operator $[A + B]$ takes the following form

$$y = [A + B]x$$

$$y(t) = g(x(o),x(1)) - Mx(o) - Nx(1) + \int_0^t [f(s,x(s)) - V(s)x(s)]\, ds$$

Let us define

$$h(x,y) = g(x,y) - Mx - Ny$$

$$F(t,x(t)) = f(t,x(t)) - V(t)\, x(t) .$$

Assume that $h(x,y)\colon D \times D \to R^n$ satisfies

$$||h(x,y) - h(u,v)||_n \leq L_{h1}||x-u||_n + L_{h2}||y-v||_n$$

and $\qquad h(0,0) = a.$

also that $F(t,x)$ satisfies all the conditions of the property (H) with f replaced by F.

From the explicit expression for the operator $[A + B]$ we see that for any $x \in C^n [D]$, y is an absolutely continuous function. In a similar way for any given absolutely continuous (AC) function y the operator B shows that x is an AC function. Note that in B operator y will always be given as an AC function, because according to the theorem 2.4.5 this will be generated by the operator $[A + B]$.

Differentiating the expression for B, which is possible because of absolute continuity of y and x, we get

$$\dot{x}(t) = V(t)\, x(t) + \dot{y}(t)$$

while putting $t = o$, we find

$$Mx(o) + Nx(1) = -y(o)$$

Now if the matrix $[\, M + N\, \Phi(1,o)\,]$ is non singular where $\Phi(t,t_o)$ is the fundamental matrix of the system

$$\dot{x}(t) = V(t)\, x(t)$$

with $\Phi(t_o,t_o) = I$, $o = t_o \leq t \leq 1$

then by Lemma 1.5.6, the inverse of the operator B can be given by

$$x = B^{-1}y \qquad x(t) = -H(t)\, y(o) + \int_o^1 G(t,s)\, \dot{y}(s)\, ds$$

where the Green's matrices $H(t)$ and $G(t,s)$, as mentioned in section 1.5, are given by

$$H(t) = \Phi(t,o) \left[M + N \Phi(1,o) \right]^{-1}$$

$$G(t,s) = H(t)M \Phi(o,s) \qquad o \leq s \leq t$$

$$= -H(t) N \Phi(1,s) \qquad t \leq s \leq 1$$

To compute the value of β for the operator B we rewrite the inverse expression as

$$x(t) = - H(t)y(o) + G(t,1)y(1) - G(t,o)y(o) + + \int_o^1 G_s(t,s)y(s)ds$$

$$= - H(t)\left[M+I \right] y(o) - H(t) N y(1) + \int_o^1 G_s(t,s)y(s)ds$$

Where

$$G_s(t,s) = \frac{d}{ds} G(t,s)$$

Assume that

$$||H(t)[I+M] ||_c \leq \beta_1, \quad ||H(t)N||_c \leq \beta_2, \quad ||G_s(t,s)||_c \leq \beta_3$$

with the above bounds we compute that

$$||x||_c \leq (\beta_1 + \beta_2 + \beta_3)||y||_c$$

and hence

$$\beta = \beta_1 + \beta_2 + \beta_3$$

To calculate the constant α for the operator $[A+B]$ we observe that, for any $x,y \in C^n[D]$,

$$\|[A+B]x(t)-[A+B]y(t)\|_n$$

$$\leq \|h(x(o),x(1))-h(y(o),y(1))\|_n + \int_o^t \|F(s,x(s))-F(s,y(s))\|_n ds$$

$$\leq L_{h1}\|x(o)-y(o)\|_n + L_{h2}\|x(1)-y(1)\|_n + L_F \int_o^t \|x(s)-y(s)\|_n ds$$

and hence

$$\|[A+B]x-[A+B]y\|_c \leq (L_{h1}+L_{h2}+L_F)\|x-y\|_c$$

therefore α can be taken as

$$\alpha = L_{h1} + L_{h2} + L_F$$

Let S_r be a closed subset of D. Then for any $x \in C^n[S_r]$ we have

$$\|[A+B]x(t)\|_n$$

$$\leq \|h(x(o),x(1))-h(o,o)\|_n + \|h(o,o)\|_n + \int_o^1 L_F\|x(s)\|_n ds$$

$$\leq L_{h1}\|x(o)\|_n + L_{h2}\|x(1)\|_n + \|a\|_n + \int_o^1 L_F\|x(s)\|_n ds$$

or

$$\|[A+B]x\|_c \leq (L_{h1}+L_{h2}+L_F)\|x\|_c + \|a\|_c$$

$$\leq \alpha r + \|a\|_c \,.$$

Thus to satisfy the condition

$$[A+B]\, S_r \subset S_{r/\beta}$$

we need

$$\alpha r + ||a||_c \leq r/\beta$$

i.e. $||a||_c \leq r(1 - \alpha\beta)/\beta$.

Hence we have proved the following theorem.

THEOREM 3.5.1

Suppose the following conditions hold

(i) The function $F(t,x)$ defined by

$$F(t,x(t)) = f(t, x(t)) - V(t)\, x(t)$$

satisfies all the conditions of the property(H) mentioned in definition 3.2.9 with f replaced by F.

(ii) There exist nxn matrices M and N such that the function $h : D \times D \to R^n$ defined by

$$h(x,y) = g(x,y) - Mx - Ny$$

is continuous and satisfies

$$||h(x,y)- h(u,v)||_n \leq L_{h1}||x-u||_n + L_{h2}||y-v||_n$$

(iii) There exists n x n matrix V(t) with integrable elements such that the matrix $[\, M+ N\Phi(1,o)\,]$ is non singular, where $\Phi(t,t_o)$ is the fundamental matrix of the system

$$\dot{x}(t) = V(t)\, x(t)$$

with $\Phi(t_o,t_o) = I, \quad o = t_o \leq t \leq 1$

(iv) The functions $H(t)$ and $G(t,s)$ as defined below

$$H(t) = \Phi(t,o)[M + N\Phi(1,o)]^{-1}$$

$$G(t,s) = H(t)\,M\,\Phi(o,s) \qquad o \leq s \leq t$$

$$= -H(t)\,N\,\Phi(1,s) \qquad t \leq s \leq 1$$

satisfy the bounds, given by

$$||H(t)[I+M]||_c \leq \beta_1 \quad ||H(t)N||_c \leq \beta_2 \quad ||G_s(t,s)||_c \leq \beta_3$$

where $G_s(t,s) = \frac{d}{ds} G(t,s)$, such that

$$\alpha\beta < 1$$

where $\alpha = L_{h1} + L_{h2} + L_F$ and $\beta = \beta_1 + \beta_2 + \beta_3$

(v) The element $h(o,o) = a$ has the bound given by

$$||a||_c \leq r(1 - \alpha\beta)/\beta$$

for some $r > o$ such that $S_r \subset D$.

Then the BVP

$$x = f(t,x(t)) \qquad g(x(o), x(1)) = 0$$

has a unique solution $\emptyset \in C^n[S_r]$.

<u>EXAMPLES</u> 3.6

In this section we present few numerical examples on the non-linear BVP. The idea is to show that the

assumptions on theorem 2.4.5. and the representation given by theorem 3.3.1 are valid and the set of problems that can be solved is nonempty.

EXAMPLE 3.6.1

It is required to find out the solution of the following problem

$$\dot{x}_1(t) = \frac{1}{4} \frac{x_2(t)}{1+x_2^2(t)}$$

$$\dot{x}_2(t) = \frac{1}{4} \frac{x_1(t)}{1+x_1^2(t)}$$

$$x_1(o) = 1 \quad x_2(1) = 1.5$$

Solution : By the theorem 3.3.1 the above problem has the equivalent representation given by

$$x_1(t) = 1 + \int_o^t \frac{1}{4} x_2(s) / (1+x_2^2(s)) \, ds$$

$$x_2(t) = x_2(o) + 1.5 - x_2(1) + \int_o^t \frac{1}{4} x_1(s)/(1+x_1^2(s)) ds$$

Thus the operator A will have the form

$$y = Ax \quad y_1(t) = 1 - x_1(t) + \frac{1}{4} \int_o^t x_2(s)/(1+x_2^2(s)) ds$$

$$y_2(t) = x_2(o) + 1.5 - x_2(1) - x_2(t) + \frac{1}{4} \int_o^t x_1(s)/(1+x_1^2(s)) ds$$

A natural choice of B is

$y = Bx \qquad y_1(t) = x_1(t)$

$$y_2(t) = x_2(t) - x_2(o) + x_2(1)$$

Thus $[A + B]$ will have the form

$$y = [A + B]x \quad y_1(t) = 1 + \frac{1}{4}\int_o^t x_2(s)/(1+ x_2^2(s))\, ds$$

$$y_2(t) = 1.5 + \frac{1}{4}\int_o^t x_1(s)/(1+x_1^2(s))ds$$

Both B and $[A + B]$ are defined over the whole space C^2. And hence the condition $[A+B]\, S_r \subset S_{r/\beta}$ will not be necessary.

The operator B has the inverse given

$x = B^{-1}y \qquad x_1(t) = y_1(t)$

$$x_2(t) = y_2(t) + y_2(o) - y_2(1) \; .$$

Then for any $y \in C^2$, we have

$$||x||_c = \max_i \sup_t ||x(t)||_n = \max_i \sup_t \begin{bmatrix} |x_1(t)| \\ |x_2(t)| \end{bmatrix}$$

$$\leq \max_i \sup_t \begin{bmatrix} |y_1(t)| \\ 3|y_2(t)| \end{bmatrix} = \max_i \sup_t \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} |y_1(t)| \\ |y_2(t)| \end{bmatrix}$$

$$\leq \left|\left| \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \right|\right| \cdot ||y||_c \leq 3\, ||y||_c$$

and hence $\beta = 3$.

Also for any $x, y \in C^2$

$$||[A+B]x - [A+B]y||_c$$

$$\leq \max_i \sup_t \begin{bmatrix} \frac{1}{4}\int_0^t |[x_2/(1+x_2^2)] - [y_2/(1+y_2^2)]|ds \\ \frac{1}{4}\int_0^t |[x_1/(1+x_1^2)] - [y_1/(1+y_1^2)]|ds \end{bmatrix}$$

$$\leq \max_i \sup_t \begin{bmatrix} \frac{1}{4}\int_0^t |x_2 - y_2|ds \\ \frac{1}{4}\int_0^t |x_1 - y_1|ds \end{bmatrix}$$

$$= \max_i \sup_t \begin{bmatrix} 0 & 1/4 \\ 1/4 & 0 \end{bmatrix} \int_0^t \begin{bmatrix} |x_1 - y_1| \\ |x_2 - y_2| \end{bmatrix} ds$$

$$\leq \frac{1}{4}\,||x - y||_c$$

and hence $\alpha = 1/4$. Thus since

$$\alpha\beta = \frac{1}{4}.3 = 3/4 < 1$$

the above problem has a unique solution.

We have the following numerical values for $x_1(t)$ and $x_2(t)$ as the solution of the BVP .

$t =$	0.0	0.2	0.4	0.6	0.8	1.0
$x_1(t)=$	1.000	1.024	1.047	1.071	1.094	1.117
$x_2(t)=$	1.375	1.400	1.425	1.450	1.475	1.500

EXAMPLE 3.6.2

Consider now the following example,

$$\dot{x}_1(t) = -x_1(t) + \frac{1}{4}\,\frac{x_2(t)}{1+x_2^2(t)} \qquad x_1(o) = 1$$

$$\dot{x}_2(t) = x_2(t) + \frac{1}{4}\,\frac{x_1(t)}{1+x_1^2(t)} \qquad x_2(1) = 1.5$$

Solution : Define the operators in the following way

$$A \quad y_1(t) = 1-x_1(t) - \int_o^t x_1(s)ds + \frac{1}{4}\int_o^t x_2(s)/(1+x_2^2(s))ds$$

$$y_2(t) = 1.5-x_2(1)+ x_2(o)- x_2(t)+ \int_o^t x_2(s)\, ds$$

$$+ \frac{1}{4}\int_o^t x_1(s) / (1+x_1^2(s))\, ds$$

$$B \quad y_1(t) = x_1(t) + \int_o^t x_1(s)\, ds$$

$$y_2(t) = x_2(t)- x_2(o) + x_2(1) - \int_o^t x_2(s)\, ds$$

$$A+B \quad y_1(t) = 1+ \frac{1}{4}\int_o^t x_2(s)/(1 + x_2^2(s))ds$$

$$y_2(t) = 1.5+ \frac{1}{4}\int_o^t x_1(s)/(1+ x_1^2(s))\, ds$$

Clearly all the operators are defined and continuous over whole of R^2. We compute the value of β and α and show

that $\alpha\beta < 1$, then theorem 2.4.5 will assure the existence and uniqueness of solution.

From the first equation of the operator B we get

$$\dot{x}_1(t) = -x_1(t) + \dot{y}_1(t) + \qquad x_1(o) = y_1(o)$$

The solution of which is given by

$$x_1(t) = e^{-t}y_1(o) + \int_o^t e^{-t+s}\,\dot{y}_1(s)ds$$

$$= y_1(t) - \int_o^t e^{-t+s}\,y_1(s)ds$$

Thus

$$|x_1|_c = \sup_t\ |x_1(t)| \leq \sup_t |y_1(t)| + \int_o^t e^{-t+s}|y_1(s)|ds$$

$$\leq \sup_t\ [\ 1+ \int_o^t e^{-t+s}\ ds\ ].|y_1|_c \leq 2|y_1|_c$$

Similarly the second equation is equivalent to

$$\dot{x}_2(t) = x_2(t) + \dot{y}_2(t) \qquad x_2(1) = y_2(o)$$

the solution of which is given by

$$x_2\ (t) = e^{t-1}\ y_2(o) + \int_1^t e^{t-s}\ \dot{y}_2(s)ds$$

$$= e^{t-1}[\ y_2(o)- y_2(1)\ ] + y_2(t)+ \int_1^t e^{t-s}y_2(s)ds$$

The above gives

$$|x_2|_c = \sup_t |x_2(t)| \leq \sup_t \left[2e^{t-1}+1+\int_1^t e^{t-s}\, ds\right].|y_2|_c$$

$$= \sup_t \left[2+ e^{t-1} \right].|y_2|_c \leq 3|y_2|_c$$

From the above two inequalities we get

$$||x||_c \leq 3||y||_c$$

and hence $\beta = 3$.

Proceeding along the lines of the problem 3.5.1, we compute the quantity α for [A+B] as 1/4, giving us

$$\alpha\beta = \frac{1}{4} \cdot 3 = 3/4 < 1$$

and hence the BVP has a unique solution.

The following table gives the numerical values of the solution

t =	0.0	0.2	0.4	0.6	0.8	1.0
$x_1(t)$ =	1.000	0.8372	0.7064	0.6007	0.5143	0.4428
$x_2(t)$ =	0.4548	0.5922	0.7589	0.9607	1.2050	1.5000

EXAMPLE 3.6.3

We solve the following problem with nonlinear integral B operator using the theorem 2.4.9.

$$\dot{x}_1(t) = \frac{x_2}{4+x_2^2} - x_1 \qquad x_1(o) = 0$$

$$\dot{x}_2(t) = \frac{x_1}{4+x_1^2} - 2x_1^2 \qquad x_2(1) = -1$$

<u>Solution</u> : In this case the operator A is defined as

$$A \quad y_1(t) = -x_1(t) + \int_o^t x_2 / 4+x_2^2 - \int_o^t x_1(s)\, ds$$

$$y_2(t) = -x_2(t) + x_2(o) + 1 + x_2(1) +$$

$$+ \int_o^t x_1(s) / (4 + x_1^2(s))ds - \int_o^t 2x_1^2(s)ds$$

Take the following non-linear B operator

$$B \quad y_1(t) = x_1(t) + \int_o^t x_1(s)\, ds$$

$$y_2(t) = x_2(t) - x_2(o) - x_2(1) + \int_o^t 2x_1^2(s)ds$$

which gives $[\, A + B \,]$ as

$$A+B \quad y_1(t) = \int_o^t x_2(s) / (4 + x_2^2(s))ds$$

$$y_2(t) = 1 + \int_o^t x_1(s) / (4 + x_1^2(s))\, ds$$

Before finding out the values of the parameters α and β

let us calculate the Lipschitz constant and the bound of the function $x/(a+x^2)$.

$$(d/dx)\ (x/(a+x^2)) = (a-x^2)/(a+x^2)^2$$

which attains the maximum value at $x = 0$. Hence the Lip Lipschitz constant is $1/a$. The function $x/(a+x^2)$ attains its maximum at $x = \sqrt{a}$ and therefore the bound is $\sqrt{a}/2a = 1/2\sqrt{a}$.

We find out the inverse of B. From the first equation of the operator B, it follows that

$$\dot{x}_1(t) = \dot{y}_1(t) - x_1(t) \qquad x_1(o) = y_1(o) = 0$$

which has the solution

$$x_1(t) = \int_o^t e^{-(t-s)}\ \dot{y}_1(s)ds$$

$y_1(o) = 0$ follows from the first equation of [A+B].

Similarly the second equation gives

$$\dot{x}_2 = \dot{y}_2(t) - 2x_1^2(t) = \dot{y}_2(t) - 2\left[\int_o^t e^{-(t-s)}\dot{y}_1(s)ds\right]^2$$

$$x_2(1) = y_2(o) = 1$$

From which we get

$$x_2(t) = 1+ y_2(t) - y_2(1) - \int_1^t 2\left[\int_0^s e^{-(s-w)} \dot{y}_1(w)dw \right]^2 ds$$

Now substitute the values of $\dot{y}_1(t)$ and $\dot{y}_2(t)$ from the operator $[\ A+B\ ]$ in the above two equations for $x_1(t)$ and $x_2(t)$ and get the following set of IEs.

$$x_1(t) = \int_0^t e^{-(t-s)} \frac{x_2(s)}{4 + x_2^2(s)} ds$$

$$x_2(t) = 1+ \int_0^t \frac{x_1}{4+ x_1^2} ds - \int_0^1 \frac{x_1}{4+x_1^2} ds$$

$$- 2 \int_1^t \left[\int_0^s e^{-(s-w)} \frac{x_2}{4+ x_2^2} dw \right]^2 ds$$

$$= 1 - \int_t^1 \frac{x_1}{4 + x_1^2} ds - 2 \int_1^t \left[\int_0^s e^{-(s-w)} \frac{x_2}{4 + x_2^2} dw\right]^2 ds$$

The solutions of the above two equations can now be studied by CMP. Since they are defined over whole of R^2 we need only to test the first condition of theorem 2.2.1 . Let us define the operator $T : C^2 \to C^2$ by the following

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (t) = \begin{bmatrix} \int_0^t e^{-(t-s)} x_2(s)/(4 + x_2^2(s))ds \\ 1 - \int_t^1 x_1/(4+ x_1^2)ds - 2 \int_1^t \left[\int_0^s e^{-(s-w)} \frac{x_2}{4+x_2^2} dw\right]^2 ds \end{bmatrix}$$

Which gives

$$||Tu - Tv||_c = \max \operatorname*{Sup}_t ||T \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}(t) - T \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}(t)||_2$$

$$\leq \max \operatorname*{Sup}_t \begin{bmatrix} \int_0^{1+} ||u_2(s)/(4+u_2^2(s)) - v_2(s)/(4+v_2^2(s))|||ds \\ \int_0^1 \left| \frac{u_1}{4+u_1^2} - \frac{v_1}{4+v_1^2} \right| ds + 2\int_0^1 || \int_0^s \frac{u_2}{4+u_2^2} + \frac{v_2}{4+v_2^2} || \\ . || \frac{u_2}{4+u_2^2} - \frac{v_2}{4+v_2^2} || dwds \end{bmatrix}$$

$$\leq \max \operatorname*{Sup}_t || \begin{bmatrix} 0 & 1/4 \\ 1/4 & 1/4 \end{bmatrix} || . \int_0^1 \begin{bmatrix} |u_1(s) - v_1(s)| \\ |u_2(s) - v_2(s)| \end{bmatrix} ds$$

$$\leq \frac{1}{2} ||u - v||_c .$$

Hence $\alpha = 1/2 < 1$ which, by CMP, shows that the operator T has a unique fixed point. In this problem the sequence $\{x^n\}$ is generated as follows:

$$x_1^{n+1}(t) = \int_0^t e^{-(t-s)} \; x_2^n(s)/(4+(x_2^n(s))^2) \; ds$$

$$x_2^{n+1}(t) = 1 - \int_t^1 \frac{x_1^n(s)}{4+(x_1^n(s))^2} ds - 2 \int_1^t [\; x_1^n(s) \;]^2 \; ds$$

starting with any arbitrary $x_1^0(t)$ and $x_2^0(t)$.

The table below gives the numerical values of the solution functions $x_1(t)$ and $x_2(t)$

t	= 0.0	0.2	0.4	0.6	0.8	1.0
$x_1(t)$	=0.0000	-0.0359	-0.0654	-0.0896	-0.1094	-0.1257
$x_2(t)$	=-.9954	-0.9960	-0.9975	-0.9988	-0.9997	-1.0000

REMARKS 3.7

It has been observed in the last section that the choice of the operator B has been dictated by the form of the operator A. In many cases it may not be so. It will usually be simpler to select a linear B operator of the form given in section 3.5. The choice of the matrices M,N and V(t) in the theorem 3.5.1 should be made in such a way that the quantities L_{h1}, L_{h2}, L_F are reduced to very small values. For example the matrix V(t) may be selected in such a way that the Frechet Derivative of the operator

$$F(t,x) = f(t,x) - V(t)x$$

over a certain region inside D is minimized which then will ensure the lowest value of L_F.

In example 3.6.1, because of the very low value of the Lipschitz constant of the function f(t,x)

it has been possible to select an algebraic B operator. This is an advantage of the representation presented here over the method discussed in [8] which will always give an integral B operator and will be dictated by the expression of the operator A.

It is interesting to observe in the above examples, that, if the element y, in the range of the operator [A+B] is substituted in the domain of B^{-1}, the resultant expression will coincide with IE representation of the BVP as has been presented in [8]. Of course this will be so only when B is linear.

The essential idea behind the fixed point theorem 2.4.5 or 2.4.9 is to extract a portion of the operator A, which prevents the direct application of CMP, and find its inverse in terms of the remaining part of A and then apply CMP. Examples given in the last section, specially the example 3.6.3 illustrates this procedure.

A little observation of the examples will show that the direct application of CMP to the operator $B^{-1}[A + B]$ will usually give less restrictive conditions on A than will be required by setting $\alpha\beta < 1$ where $\beta = ||B^{-1}||$ and α = Lipschitz constant of $[A + B]$. This conclusion is the converse of the one observed in [8].

CHAPTER - FOUR

CONTROLLABILITY PROBLEM

INTRODUCTION 4.1

Formally speaking a system $\dot{x} = f(x,u)$ defined over the real space, where f,x,u are suitable vectors, is said to be controllable, if there exists a control function u , which steers the system from any given $x(t_0) = x_0$ to any given $x(t_1) = x_1$. Ever since the notion of controllability was introduced by Kallman, various authors have worked on this problem in various different ways and a lot of materials are available in this direction. Following is a brief discussion of some of the interesting literature.

Controllability for linear systems is well developed, any standard text book on control theory may be consulted for this result. In [23] a good chapter has been presented for the study of non-linear controllability problems. In one method, it has been shown that the domain of null controllability, i.e. the set of points that can be steered to the origin, is an open set under the assumption that the system when linearized at origin is controllable. Another result

which gives global controllability, makes use of the Lyapunov function and the controllability of the system linearized at the origin. In both of the above results it has been assumed that the control is restricted in some subset of the underlying space.

Some results are available regarding the use of fixed point theorems for the study of controllability properties. In [24] Schauder's fixed point theorem has been used for the study of

$$\dot{x} = A(t,x)\, x + B(t,x)\, u$$

by representing it in the form of a suitable integral equation, which makes use of the corresponding linear system $\dot{x} = A(t,z)x + B(t,z)u$ for some z belonging to certain class of functions. Global result has been obtained under the assumption

$$|a_{ij}(t,x)| \leq M \quad |b_{ij}(t,x)| \leq N$$

for all t, x, where a_{ij} and b_{ij} are elements of the matrices $A(t,x)$ and $B(t,x)$ respectively.

The controllability property of

$\dot{x} = g(t,x) + k(t,u)$ has been investigated in an interesting paper [25] with the help of fixed point

theorem of Ky Fan [26]. With some convexity assumptions on the domain of $\dot{x}(t)$ controllability results were obtained, while without this assumption an ϵ - controllability has been derived which assures that the final state will be within any given ϵ- sphere around the required given state.

The above methods are essentially qualitative methods, in the sense that they do not give any procedure to compute a control that will be required to steer the system. In [27], a constructive method has been presented for finding out the steering control of the system $\dot{x} = \theta(x) + Q(t)$ where $Q(t)$ is a vector with only one non-zero element. The method generates a sequence of controls by successive use of the equivalent integral equation representation of the linear system $\dot{x} = A(t)x + f(t)$ and non-linear system $\dot{x} = \theta(x) - A(t)x + Q(t)$, which converges to the required control. The control is selected from the class of piecewise constant functions over a given control interval.

In this chapter, we present a study of the controllability property of the system $\dot{x} = f(t,x,u)$ by the help of the fixed point theorem presented in chapter two. The controllability problem which is a

kind of boundary value problem has been represented by an equivalent integral equation and then investigated. An example has been included to illustrate the procedure.

DEFINITION 4.2

Let x and y be two vectors, then $[xy]$ will denote a vector each component of which is the product of the corresponding components of the vectors x and y, $[xx]$ will be denoted by $[x]^2$. We follow the notations of section 3.2 regarding the norms etc.

DEFINITION 4.2.1

The function f will be said to satisfy the property (PP) if the following condition hold

(i) $f(t,x,u) : I \times D \times U \to R^n$ where I, D, U are open subsets of R, R^n and R^m respectively, with $[0,1] \subset I$.

(ii) $f(t,x,u)$ is continuous in $(x,u) \in D \times U$ for each fixed $t \in I$ and is measurable in $t \in I$ for each fixed $(x,u) \in D \times U$.

(iii) There exists a Lebesgue integrable function $m(t)$ such that

$||f(t,x,u)||_n \leq m(t)$ a.e. in I and for all $(x,u) \in D \times U$.

(iv) f satisfies the uniform Lipschitz condition in $D \times U$,

$$||f(t,x,u) - f(t,y,v)||_n \leq L_1||x-y||_n + L_2||u - v||_n$$

for almost all $t \in I$.

(v) $f(t,0,0) = 0$ a.e. in I.

Throughout in this chapter we will consider the following dynamical system

$$\dot{x}(t) = f(t,x(t), u(t)) \qquad (4.2.2)$$

DEFINITION 4.2.3

A bounded measurable function u with values in U will be called an admissible control. By trajectory we will mean any absolutely continuous (AC) function θ that satisfies the system 4.2.2. with some admissible control. $\theta(t)$ will denote the state of the system at time t.

DEFINITION 4.2.4

The system (4.2.2) is said to be controllable if for any given set $\{[t_0,t_1] \subset I,\ a, b \in D\}$, there exists an admissible control such that the

trajectory of (4.2.2) satisfies the condition

$$x(t_o) = a \qquad x(t_1) = b \tag{4.2.5}$$

For the sake of simplicity and without loss of any generality we will assume that $[t_o, t_1] = [o, 1]$. From now onwards by controllability problem we will mean the following Boundary Value Problem

$$\dot{x}(t) = f(t,x(t), u(t)) \tag{4.2.6}$$

$$x(o) = a \qquad x(1) = b \tag{4.2.7}$$

For some a,b belonging to D .

REPRESENTATION 4.3

THEOREM 4.3.1

Let $f(t,x,u)$ satisfy the first three condition of the property (PP). Let K be an $m \times n$ matrix such that $K[x]^2 = 0 \implies x = 0$ for any vector $x \in D$. Then any solution of the following IEs

$$x(t) = a + \int_0^t f(s,x(s),u(s))ds \tag{4.3.2}$$

$$u(t) = u(1) + K\left[a-b+\int_0^t f(s,x(s),u(s))ds \right]^2 \tag{4.3.3}$$

is a solution of the controllability problem (4.2.6) and (4.2.7).

Proof : Let $v \in C^m[U]$ and $\theta \in C^n[D]$ satisfy the system of equations (4.3.2) and (4.3.3). Then (4.3.2) gives

$$\theta(t) = a + \int_0^t f(s, \theta(s), v(s))\, ds \qquad (4.3.4)$$

Clearly θ is an AC function, hence by differentiating, we get

$$\dot{\theta}(t) = f(t, \theta(t), v(t)).$$

and at $t = 0$, (4.3.4) gives

$$\theta(0) = a.$$

Setting $t = 1$, in (4.3.3), we get

$$v(1) = v(1) + K\left[a-b + \int_0^1 f(s,\theta(s), v(s))ds \right]^2$$

$$\text{i.e.} \quad K\left[a-b + \int_0^1 f(s, \theta(s), v(s))ds \right]^2 = 0$$

which, by hypothesis on K, gives

$$a + \int_0^1 f(s,\theta(s), v(s))ds = b \quad .$$

But because of (4.3.4) we see that the left hand side of the above is equal to $\theta(1)$, and hence

$$\theta(1) = b$$

Thus θ and v satisfy (4.2.6) and (4.2.7).

Q.E.D.

SOLUTION PROBLEM 4.4

Theorem 4.3.1 shows that instead of studying the controllability problem defined by (4.2.6) and (4.2.7) we can study the IEs (4.3.2) and (4.3.3). Let us define the operators A and B by

$$A \quad y_1(t) = a - x(t) + \int_0^t f(s,x(s),u(s))ds$$

$$y_2(t) = u(1) - u(t) + K\left[a-b+ \int_0^t f(s,x(s),u(s))ds \right]^2$$

$$B \quad y_1(t) = x(t) - \int_0^t V_1(s)\, x(s)ds - \int_0^t W_1(s)\, u(s)ds$$

$$y_2(t) = u(t)-u(1)- K \int_0^t V_2(s)x(s)ds - K \int_0^t W_2(s)u(s)ds$$

Hence the operator [A+B] will have the form

$$A+B \quad y_1(t) = a + \int_0^t \left[f(s,x,u) - V_1\, x - W_1 u \right]ds$$

$$y_2(t) = K\left\{ \left[a-b + \int_0^t f(s,x,u)ds \right]^2 - \int_0^t V_2 x\, ds - \int_0^t W_2 u ds\right\}$$

With the operators defined as above, we can proceed along the lines of section 3.5 to conclude a theorem like the one presented there. Following theorem is presented for a simplified case when x and u have the same dimensions.

Let us denote by I_n and 0_n respectively the identity and the null matrices of size nxn.

THEOREM 4.4.1

Let $f(t,x,u)$ be a mapping of R^{1+n+n} into R^n and a,b be any two elements in R^n. Suppose that the following conditions are satisfied

(i) There exists two nxn matrices $V(t)$ and $W(t)$ with the norms bounded by integrable functions such that the matrix $[\, M + N\,\Phi(1,o)\,]$ is non-singular. Where

$$M = I_{2n} \quad , \quad N = \begin{bmatrix} 0_n & 0_n \\ 0_n & -I_n \end{bmatrix}$$

and $\Phi(t,t_o)$ is the fundamental matrix of the system

$$\dot{z}(t) = P(t)\; z(t) \qquad P(t) = \begin{bmatrix} V(t) & W(t) \\ V(t) & W(t) \end{bmatrix}$$

with $\Phi(t_o,t_o) = I_{2n}$, $t_o = o$.

(ii) The function F defined by

$$F(t,x,u) = f(t,x,u) - V(t)x - W(t)u$$

satisfies the property (PP) with f replaced by F and I,D,U by R, R^n, R^n respectively.

(iii) The Green's matrix $G(t,s)$ defined by

$$G(t,s) = \Phi(t,o)[\, M+N\ \Phi(1,o)\,]^{-1} M\ \Phi(o,s) \qquad o \leq s \leq t$$

$$= \Phi(t,o)[\, M+N\ \Phi(1,o)\,]^{-1} N\ \Phi(1,s) \qquad t \leq s \leq 1$$

has the norm δ , i.e. $||G||_c = \delta$, such that

$$\delta(L_1 + L_2) < 1$$

Then the controllability problem defined by

$$\dot{x}(t) = f(t,x(t),u(t)) \qquad x(o) = a\ ,\ x(1) = b$$

has a solution.

<u>Proof</u> : Suppose that the assumptions hold. By theorem 4.3.1, the above controllability problem can be solved using the IEs

$$x(t) = a + \int_o^t f(s,x(s),\ u(s))ds$$

$$u(t) = u(1) + a-b + \int_o^t f(s,x(s),\ u(s))ds\ .$$

Accordingly let us define the operator A by

$$A \quad y_1(t) = a-x(t) + \int_o^t f(s,x(s),u(s))ds$$

$$y_2(t) = a-b + u(1) - u(t) + \int_o^t f(s,x(s),u(s))\ ds$$

and select the operator B as

$$B \quad y_1(t) = x(t) - \int_0^t V(s)\ x(s)\ ds - \int_0^t W(s)\ u(s)\ ds$$

$$y_2(t) = u(t) - u(1) - \int_0^t V(s)\ x(s) ds - \int_0^t W(s)\ u(s)\ ds$$

Then the operator [A+B] takes the form

$$A+B \quad y_1(t) = a + \int_0^t F(s,x(s),\ u(s)) ds$$

$$y_2(t) = a-b + \int_0^t F(s,x(s),u(s)) ds$$

It is easy to see that the operator B is equivalent to the following DE

$$\begin{bmatrix} \dot{x}(t) \\ \dot{u}(t) \end{bmatrix} = \begin{bmatrix} V(t) & W(t) \\ V(t) & W(t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix}$$

with the BC

$$\begin{bmatrix} I_n & O_n \\ O_n & I_n \end{bmatrix} \begin{bmatrix} x(o) \\ u(o) \end{bmatrix} + \begin{bmatrix} O_n & O_n \\ O_n & -I_n \end{bmatrix} \begin{bmatrix} x(1) \\ u(1) \end{bmatrix} = \begin{bmatrix} y_1(o) \\ y_2(o) \end{bmatrix}$$

The solution of which is given by

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = H(t) \begin{bmatrix} y_1(o) \\ y_2(o) \end{bmatrix} + \int_0^1 G(t,s) \begin{bmatrix} \dot{y}_1(s) \\ \dot{y}_2(s) \end{bmatrix} ds$$

where $H(t) = \Phi(t,o)\ [\ M + N\ \Phi(1,o)\ ]^{-1}$.

Substituting the value of $y_1(t)$ and $y_2(t)$ from the expressions for the operator [A+B] in the above IE, we get a new operator T given by

$$T \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = H(t) \begin{bmatrix} a \\ a-b \end{bmatrix} + \int_0^1 G(t,s) \begin{bmatrix} F(s,x(s),u(s)) \\ F(s,x(s),u(s)) \end{bmatrix} ds \tag{4.4.2}$$

Clearly T is a mapping from C^{2n} into C^{2n}, satisfying the Lipschitz condition given by

$$||Tp - Tq||_c \leq \delta(L_1 + L_2)||p - q||_c$$

where $p,q \in C^{2n}$. Since $\delta(L_1+L_2) < 1$, by CMP T has a fixed point. Therefore by theorem 2.4.9, because T is nothing but $B^{-1}[\ A + B\ ]$ there exists (x^*, u^*) such that

$$A \begin{bmatrix} x^* \\ u^* \end{bmatrix} = 0$$

Q.E.D.

REMARKS 4.4.3

(i) As has been mentioned before, the advantage of the method used here is that it will help us to find out the control function numerically. The method presented here finds out only one unique control, whereas in reality there may be many controls that can be used to

steer the system. This has happened because the IEs used, to find out the control, puts an equality constraint on the control variable.

(ii) The idea of augmenting the IE (4.3.2) by (4.3.3) is to get a composite operator mapping C^{m+n} into itself, thus removing the difficulty in handling IE (4.3.2) which maps C^{m+n} into C^n.

(iii) The IE (4.3.3) may be replaced by its most general form $0 = h(u(t), t) + g(x(t), b)$ where $h: U \times I \to R^m$, $g : D \times D \to R^m$ and 0 is the null element of R^m, with the assumptions that

$$h(u(1),1) = 0 \quad \text{and}$$

$$g(x(1), b) = 0 \Rightarrow x(1) = b \ .$$

However it appears that the particular form presented in theorem 4.3.1 is most suitable for general problems.

<u>EXAMPLE</u> 4.5

We present the following example illustrating the application of theorem 2.4.5 in controllability problem.

$$\dot{x}(t) = \epsilon \frac{x(t)}{1+ x^2(t)} + \epsilon \frac{u(t)}{1+ u^2(t)} - u(t)$$

$$x(o) = a \qquad\qquad x(1) = 0 \ .$$

Solution : Theorem 4.3.1 gives the following IE representation of the above problem.

$$x(t) = a + \int_0^t \epsilon \frac{x}{1 + x^2} ds + \int_0^t \epsilon \frac{u}{1 + u^2} ds - \int_0^t u(s)ds$$

$$u(t) = u(1) + a + \int_0^t \epsilon \frac{x}{1+x^2} ds + \int_0^t \epsilon \frac{u}{1+u^2} ds - \int_0^t u(s)ds$$

Hence we can define the operator A as

$$A \quad y_1(t) = a - x(t) + \int_0^t (\epsilon \frac{x}{1+x^2} + \epsilon \frac{u}{1+ u^2} - u) ds$$

$$y_2(t) = u(1) - u(t) + a + \int_0^t (\epsilon \frac{x}{1+ x^2} + \epsilon \frac{u}{1+u^2} - u)ds$$

Select the following B operator

$$B \quad y_1(t) = x(t) + \int_0^t u(s) ds$$

$$y_2(t) = u(t) - u(1) + \int_0^t u(s)ds$$

Thus the operator [A + B] takes the form

$$A+B \quad y_1(t) = a + \int_0^t \epsilon \frac{x}{1+x^2} + \epsilon \frac{u}{1+u^2} ds$$

$$y_2(t) = a + \int_0^t \epsilon \frac{x}{1+x^2} + \epsilon \frac{u}{1+u^2} ds$$

To compute the inverse of B we observe that B is

equivalent to

$$(d/dt)\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} o & -1 \\ o & -1 \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix}$$

with $$\begin{bmatrix} 1 & o \\ o & 1 \end{bmatrix}\begin{bmatrix} x(o) \\ u(o) \end{bmatrix} + \begin{bmatrix} o & o \\ o & -1 \end{bmatrix}\begin{bmatrix} x(1) \\ u(1) \end{bmatrix} = \begin{bmatrix} y_1(o) \\ y_2(o) \end{bmatrix}$$

The fundamental matrix corresponding to the matrix

$$P = \begin{bmatrix} o & -1 \\ o & -1 \end{bmatrix} \quad \text{is} \quad \Phi(t,t_o) = e^{(t-t_o)P} = \begin{bmatrix} 1 & e^{-(t-t_o)} - 1 \\ o & e^{-(t-t_o)} \end{bmatrix}$$

The Green's matrices are given by

$$H(t) = \Phi(t,o)\left[M + N\,\Phi(1,o) \right]^{-1}$$

$$= \begin{bmatrix} 1 & e^{1-t}(1-e^{t})/(e-1) \\ o & e^{1-t}/(e-1) \end{bmatrix}$$

$$G(t,s) = \begin{cases} G_1(t,s) = H(t)\,M\,\Phi(o,s) & o \leq s \leq t \\ G_2(t,s) = -H(t)\,N\,\Phi(1,s) & t \leq s \leq 1 \end{cases}$$

where

$$G_1(t,s) = \begin{bmatrix} 1 & (e^{s}-1) + e^{1-t+s}.(1-e^{t})/(e-1) \\ o & e^{1-t+s}/(e-1) \end{bmatrix}$$

and $G_2(t,s) = \begin{bmatrix} o & e^{s-t}(1-e^t)/(e-1) \\ o & e^{s-t}/(e-1) \end{bmatrix}$

Thus the solution is given by

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = H(t) \begin{bmatrix} y_1(o) \\ y_2(o) \end{bmatrix} + \int_o^1 G(t,s) \begin{bmatrix} \dot{y}_1(s) \\ \dot{y}_2(s) \end{bmatrix} ds$$

$$= G(t,1) \begin{bmatrix} y_1(1) \\ y_2(1) \end{bmatrix} - \int_o^1 G_s(t,s) \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} ds + \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

where we have used the fact that $G(t,o) = H(t)$

It can be easily shown that

$||G(t,1)|| \leq e$ $\qquad$ $||G_s(t,s)|| \leq e$

for all $t \in [o,1]$ $\qquad$ for all $t,s \in [o,1]$

With the above two estimates we find that the value of $\beta = 2e + 1$. From the expressions of $[A + B]$ we can easily find that its Lipschitz constant $\alpha = 2\epsilon$. Since $[A + B]$ is defined over the whole space, it can be concluded that the system can be steered from any $x(o) = a \in R$ to the origin in the time interval $[o,1]$ provided ϵ is such that

$$2\epsilon\,(2e + 1) < 1$$

The table below gives the result of a sample calculation for for $\epsilon = 0.02$ and $a = 1$

t =	0.0	0.2	0.4	0.6	0.8	1.0
$x(t)$ =	1.0000	0.7187	0.4863	0.2935	0.1333	0.0000
$u(t)$ =	1.5670	1.2900	1.0640	0.8780	0.7244	0.5974

An interesting simpler result can be obtained if we select the IE representation of the following form

$$x(t) = a + \int_0^t \epsilon \frac{x}{1+x^2}\, ds + \int_0^t \epsilon \frac{u}{1+u^2}\, ds - \int_0^t u(s)ds$$

$$0 = (1-t)a + a + \int_0^t \epsilon \frac{x}{1+x^2}\, ds + \int_0^t \epsilon \frac{u}{1+u^2}\, ds - \int_0^t u(s)ds.$$

Define the operators in the following way

A $\quad y_1(t) = a-x(t) + \int_0^t \epsilon \frac{x}{1+x^2} + \epsilon \frac{u}{1+u^2} - u\, ds$

$$y_2(t) = (1-t)a + a + \int_0^t \epsilon \frac{x}{1+x^2} + \epsilon \frac{u}{1+u^2} - u\, ds$$

B $\quad y_1(t) = x(t) + \int_0^t u(s)\, ds$

$$y_2(t) = (1-t)a + \int_0^t u(s)\, ds$$

$$A+B \quad y_1(t) = a + \int_0^t \epsilon \frac{x}{1+x^2} + \epsilon \frac{u}{1+u^2} \, ds$$

$$y_2(t) = a + \int_0^t \epsilon \cdot \frac{x}{1 + x^2} + \epsilon \frac{u}{1 + u^2} \, ds$$

From the expressions of B we get

$$u(t) = a + \dot{y}_2(t)$$

$$x(t) = y_1(t) - \int_0^t u(s)ds = y_1(t) + (1-t)a - y_2(t)$$

Substituting the values of $y_1(t)$ and $y_2(t)$ from the expressions of $[A + B]$ we get

$$u(t) = a + \epsilon \frac{x(t)}{1 + x^2(t)} + \epsilon \frac{u(t)}{1 + u^2(t)}$$

$$x(t) = (1-t)a$$

The equation for $u(t)$ will have solution if $\epsilon < 1$. Note that the trajectory is a straight line joining the point a with the origin.

Following table gives the control for $a = 1/2$ and $\epsilon = 1/4$.

t =	0.0	0.2	0.4	0.6	0.8	1.0
$u(t)$ =	0.7185	0.7039	0.6854	0.6632	0.6381	0.6112

CHAPTER - FIVE

STABILITY PROBLEM

INTRODUCTION 5.1

In this chapter we study the following dynamical system

$$\dot{x}(t) = f(t, x(t), u(t))$$

the variables, t,x,u, always real, will be called as time, state and control respectively. We call the control u, as open loop control when u is dependent only on t, and as close loop control when it is a function of both t and x.

We present the stability properties of the above system with closed loop control using Banach's Fixed point theorem, and with open loop control using a modification [29] of Kakutani's fixed point theorem[3].

Fixed point theorems of Banach, Schauder have been used [31] for the investigation of stability properties. A good number of interesting results may be obtained in [28]. In [30] Tychonoff's theorem has been used with the help of some differential inequalities. In all these cases the system considered was unforced i.e. without the control function. We

study a more general problem in some what different way.

So far as the knowledge of the author goes, theorems, like Kakutani's, have not been used for the solution of stability problems. However Ky Fan's theorem [26], which is a generalization of Kakutani's theorem in locally convex linear topological spaces has been used [25] for the investigation of the controllability problems. One difference we have here in our method is that we do not require any convexity condition on the system. The method presented here can easily be, without very little change, extended to the case of closed loop control.

<u>DEFINITION</u> 5.2

Consider the DE

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad (5.2.1)$$

with $f(t,0,0) = 0$,

defined over the real spaces, where f, x, u are suitable vectors. Under the above condition the system (5.2.1) admits a trivial solution, which is identically equal to zero, we call it as null solution.

The null solution, under certain control function u, is said to be

(i) g- stable, if there exists a $\delta > o$ such that for any $||x(t_o)|| < \delta$, the solution of (5.2.1) satisfies

$$||x(t)|| \leq g(t)$$

for all $t \geq t_o$ & for some given non-negative continuous function $g(t)$.

Clearly the above definition coincides with that of Lyapunov, if the above is true for any constant $g(t) = \epsilon > o$.

(ii) quasi-stable if for any given $\epsilon > o$, there exist $\delta > o$ and some $t_1 \geq t_o \geq o$ such that $||x(t)|| \leq \epsilon$ for all $t \geq t_1$ and $||x(t_o)|| \leq \delta$.

The above definition can be considered as g-stable with g given by

$$\begin{aligned} g(t) &= r & t_o \leq t \leq t_1 - w \\ &= r - \frac{r-\epsilon}{w}(t-t_1+r), & t_1-w \leq t \leq t_1 \\ &= \epsilon & t \geq t_1 \end{aligned}$$

where r is some admissible upper bound and w any real number greater than zero.

(iii) asymptotically stable if it is g-stable with some g asymptotic to zero.

(iv) exponentially asymptotically stable if there exist $\alpha > 0$, $\beta > 0$ such that the system is g-stable with g given by

$$g(t) = \beta\, e^{-\alpha(t-t_0)} \qquad t \geq t_0$$

(v) g_T stable if $\exists\ \delta > 0$ such that for any $||x\,(t_0)|| \leq \delta$ the solution satisfies

$$||x(t)|| \leq g(t) \quad \text{for } \forall\ t \in [\, t_0, T\,]$$

for some non-negative continuous function g.

In most of the practical problems it is sufficient to have g_T stability, when T is large. In some cases to avoid mathematical complicacies we will only study g_T stability for arbitrarily large T but not equal to infinity.

In this chapter we use the following norms. For any element $x \in R^n$ we denote the norm by

$$||x|| = [\sum_{i=1}^{n} |x_i|^2\,]^{1/2}$$

i.e. the Euclidean norm. For any element $\phi \in C^n$, we define the norm by

$$||\phi||_c = \sup_t ||\,\phi(t)||$$

provided the supremum exists. And let the norm on the operators $A(t)$ and $G(t,s)$ etc be the one generated by the corresponding Euclidean norm on Domain and range spaces. We represent these norms as before by $||A||_c$ and $||G||_c$ respectively.

CLOSED LOOP STABILITY 5.3

Let $\Phi(t,t_o)$ be the fundamental matrix of the linear system

$$\dot{x}(t) = A(t)\ x(t) \tag{5.3.1}$$

with $\Phi(t_o,t_o) = I$, $t_o = o$, where $A(t)$ is an n x n matrix with intregable elements. Let $\partial[\ A(t)\ ]$ denote the largest eigen value of the matrix

$$(1/2)\ [\ A(t) + A^T(t)\ ]$$

Then we have the following lemma [32].

LEMMA 5.3.2

Let the norm for vectors in R^n be the Euclidean norm and that on $\Phi(t,t_o)$ be the one generated by it, then

$$||\Phi(t,t_o)|| \leq \exp \int_{t_o}^{t} \partial[\ A(s)\ ]ds \tag{5.3.3}$$

THEOREM 5.3.4

Let the following conditions be satisfied

(i) There exists an nxn matrix $A(t)$ with bounded

and measurable functions as elements such that

$$\partial[\, A(t)\,] \leq -\alpha \quad , \quad \alpha > o$$

(ii) There exists a function $V: I \times D \to U$ satisfying the condition (i) and (iv) of property (H) mentioned in definition 3.2.9, where I,D and U are open subsets of R, R^n and R^m respectively, such that the function F defined by

$$F(t,x) = f(t,x(t), V(t,x)) - A(t)\, x(t)$$

satisfies the property (H) with f replaced by F

(iii) There exist a continuous bounded non-negative function $g(t)$ and a $\delta > o$ such that

$$o < \delta \leq g(t)\, e^{\alpha t} - L_F \int_o^t g(s)\, e^{\alpha s}\, ds$$

Then if $L_F < \alpha$, the system (5.2.1) is g-stable, for any $x(t_o) = x_o \in R^n$, such that

$$||x_o|| \leq \delta$$

with the control $u = V(t,x)$.

Proof : Assume that the conditions of the theorem hold and let x_o be such that $||x_o|| \leq \delta$. The system (5.2.1) is clearly equivalent to

$$\dot{x}(t) = A(t)\, x(t) + F(t,x(t))$$

where $F(t,x(t))$ is as mentioned in the hypothesis (ii), which in turn is equivalent to

$$x(t) = \phi(t,t_o)\, x_o + \int_{t_o}^{t} \phi(t,s)\, F(s,x(s))\, ds$$

The g-stability of the above IE can be studied by studying the fixed point of the operator T given by

$$y = Tx \qquad y(t) = \phi(t,t_o)x_o + \int_{t_o}^{t} \phi(t,s)F(s,x(s))ds$$

over the class of functions defined by

$$Q = \{\theta : \theta \in C^n ,\ ||\theta(t)|| \leq g(t)\}$$

For any x and $y \in Q$, we have

$$||Tx(t) - Ty(t)|| \leq \int_{t_o}^{t} ||\phi(t,s)||.||F(s,x(s))-F(s,y(s))||ds$$

$$\leq \int_{t_o}^{t} e^{-\alpha(t-s)} L_F||x(s) - y(s)||\, ds$$

Hence

$$||Tx - Ty||_c \leq L_F||x - y||_c \int_{0}^{t} e^{-\alpha(t-s)} ds \leq \frac{L_F}{\alpha}||x-y||_c$$

which shows, since $(L_F/\alpha) < 1$, that T is contraction.

Also

$$||Tx(t)|| \leq ||\phi(t,t_o)||\ ||x_o|| + \int_{t_o}^{t} ||\phi(t,s)|| \times ||F(s,x(s)||ds$$

$$\leq e^{-\alpha t}\,\delta + \int_0^t e^{-\alpha(t-s)}\, L_F||x(s)||ds$$

$$\leq e^{-\alpha t}\,\delta + L_F\, e^{-\alpha t}\int_0^t e^{\alpha s}\, g(s)\, ds$$

Substituting the expression for δ from condition (iii) we get

$$||Tx(t)|| \leq g(t)$$

thus $Tx \subset Q$. So we see that T satisfies all the conditions of Banach's Fixed point theorem 2.2.1. over the set Q. And hence there exists a $\emptyset \in Q$ such that for the control function $u = V(t, \emptyset(t))$ the system (5.2.1) is g-stable for any initial condition $x(t_o) = x_o$ with $||x_o|| \leq \delta$.

Q.E.D.

COROLLARY 5.3.5

Under the hypotheses of the theorem 5.3.4 the system is stable in the sense of Lyapunov.

Proof : We only need to show that with $g(t) = \epsilon > o$, we can find out a $\delta > o$ such that the condition (iii) will be satisfied. Now

$$g(t)\, e^{\alpha t} - L_F\int_o^t g(s)\, e^{\alpha s}\, ds = \epsilon\, e^{\alpha t} - L_F\int_o^t \epsilon\, e^{\alpha s} ds$$

$$= \epsilon\, e^{\alpha t} - (L_F/\alpha)\, e^{\alpha t} + (L_F/\alpha)\epsilon = \epsilon\left[1 - \frac{L_F}{\alpha}\right] e^{\alpha t} + (L_F/\alpha)\epsilon$$

The above shows that the condition (iii) will be satisfied if we take any $\delta > o$ satisfying $\delta \leq \epsilon$.

Q.E.D.

CORCLLARY 5.3.6

Under the assumptions of the theorem 5.3.4 the system (5.2.1) is exponentially stable.

Proof: We proceed along the lines of the proof of Corollary 5.3.5. Since $o \leq (L/\alpha) < 1$ there exists a $\beta > o$,such that $o \leq (L/(\alpha-\beta) < 1$. Let β be any such number and

$$g(t) = r\, e^{-\beta t}$$

for any $r > o$. Then

$$g(t)\, e^{\alpha t} - L_F \int_o^t g(s) e^{\alpha s}\, ds = r\, e^{(\alpha-\beta)t} - L_F r \int_o^t e^{(\alpha-\beta)s}\, ds$$

$$= r e^{(\alpha-\beta)t} - \frac{L_F r}{\alpha - \beta}\left[e^{(\alpha-\beta)t} - 1\right] = \frac{L_F r}{\alpha-\beta} + e^{(\alpha-\beta)t} r\left[1 - \frac{L_F}{\alpha-\beta}\right]$$

The above will be satisfied if $\delta \leq r$. Q.E.D.

BASIC RESULTS 5.4

In this section we present few mathematical preliminaries that will be required in the next section. Details of these materials can be found in [3].

DEFINITION 5.4.1

Let Γ be a mapping of a topological space X into a topological space Y and let x_0 be a point of X. We say that Γ is lower semicontinuous (l.s.c.) at x_0 if for each open set G meeting Γx_0 there is a neighbourhood $U(x_0)$ such that

$$x \in U(x_0) \Rightarrow \Gamma x \cap G \neq 0 .$$

DEFINITION 5.4.2

The mapping Γ is upper semicontinuous (u.s.c.) at x_0 if for each closed set G containing Γx_0, there exists a closed neighbourhood $U(x_0)$ such that

$$x \in U(x_0) \Rightarrow \Gamma x \subset G$$

The above can be rewritten also as follows. For any given $\epsilon > 0$, there exists a $\delta > 0$ such that $\Gamma x \subset [\Gamma x_0]_\epsilon$ whenever $x \in [x_0]_\delta$ where $[A]_r$ denotes the closed r-neighbourhood of the set A.

We say that the mapping Γ is u.s.c. (l.s.c.) in X if it is u.s.c. (l.s.c.) at each point of $x \in X$. Γ is continuous if it is both u.s.c. and l.s.c.

DEFINITION 5.4.3

For any set B the lower inverse of Γ is defined by

$$\Gamma^{-}B = \{x : x \in X, \quad \Gamma x \cap B \neq 0\}$$

Similarly the upper inverse is given by

$$\Gamma^{+} B = \{x : x \in X, \Gamma x \subset B\}$$

<u>THEOREM</u> 5.4.4.

A necessary and sufficient condition for Γ to be u.s.c. is that the set Γx is compact for each x and for each open set B in Y the set $\Gamma^{+}B$ is open.

<u>DEFINITION</u> 5.4.5

The mapping Γ is said to be closed if $x_n \to x_o$, $y_n \to y$, and for all n, $y_n \in \Gamma x_n$ implies that $y_o \in \Gamma x_o$.

<u>THEOREM</u> 5.4.6

Every u.s.c. mapping is closed.

<u>Proof</u> : Let $\epsilon > o$ be given. Then by u.s.c. of Γ there exists a $\delta > o$ such that for any $x \in [x_o]_\delta$ $\Gamma x \subset [\Gamma x_o]_\epsilon$. Now since $x_n \to x_o$ for all sufficiently large n, $x_n \in [x_o]_\delta$ which implies $\Gamma x_n \subset [\Gamma x_o]_\epsilon$. By theorem 5.4.4 Γx_o is compact, and therefore closed. So, since ϵ is arbitrary and Γx_o is closed, $\Gamma x_n \subset \Gamma x_o$ for all sufficiently large n. But $y_n \in \Gamma x_n$ therefore

$y_n \in \Gamma x_o$. Since $y_n \to y_o$, the above implies $y_o \in \Gamma x_o$.

Q.E.D.

LEMMA 5.4.7
(Ascoli)

Any set of uniformly bounded, equi-continuous functions defined over a compact interval is compact.

By definition, equi-continuity of a set of functions means that, for any given $\epsilon > o$ if there exists a $\delta > o$ such that, for all f in the set following holds

$$||f(t) - f(\bar{t})||_n \leq \epsilon \quad \text{whenever } |t - \bar{t}| \leq \delta$$

LEMMA 5.4.8
(Fillippov[33])

Let the vector function f(t,u) be continuous, let the set Q(t) be closed, bounded and u.s.c. with respect to inclusion in t, let the vector f(t,u) describe a set R(t) when the vector u describes the set Q(t).and let y(t) be a measurable vector function such that $y(t) \in R(t)$. Then there exists a measurable function u(t) such that f(t,u(t))=y(t) for almost all t.

A more general version of the above lemma can be found in [34]. Following is a generalization of Kakutani's

fixed point theorem to Banach spaces given by Bhonenblust and Karlin [29].

THEOREM 5.4.9

(Bhonenblust, Karlin)

Let S be a convex closed set of a Banach space. To each point x of S a non-void set $\Gamma x \subset S$ is assumed given. If

(i) $x_n \to x$, $y_n \to y$, for $\forall n$ $y_n \in \Gamma x_n$ imply $y \in \Gamma x$

(ii) the union $\bigcup \Gamma x$ over all $x \in S$ is contained in some sequentially compact set T,

then there exists a point $x^* \in S$ such that $x^* \in \Gamma x^*$

OPEN LOOP STABILITY 5.5

Let g(t) be a bounded positive continuous function defined over [o, T] where T is any positive real number less than infinity. Let θ denote the class of all Lipschitz continuous function defined over [o,T], with unformly bounded Lipschitz constant M, with values in $S_{r_i} \subset D \subset R^n$, where D is an open set. The value for M has been derived in what follows. Let the set B be defined as

$$B = \{ x : x \in \theta, \ ||x(t)|| \leq g(t), \ t \in [o, T] \} \quad (5.5.1)$$

LEMMA 5.5.2

The set B as defined in (5.5.1) is convex and compact.

Proof: Let $x, y \in B$, $w \in [0,1]$ and let

$z(t) = wx(t) + (1-w)\, y(t)$,

$$||z(t) - z(\bar{t})|| \leq w||x(t) - x(\bar{t})|| + (1-w)||y(t)-y(\bar{t})||$$

$$\leq w\, M|t - \bar{t}| + (1-w)\, M|t - \bar{t}| = M|t - \bar{t}|$$

Thus z is Lipschitz continuous with the same Lipschitz constant M and hence $z \in \theta$

$$||z(t)|| = ||wx(t) + (1-w)y(t)|| \leq w||x(t)|| + (1-w)||y(t)||$$

$$\leq wg(t) + (1-w)g(t) = g(t) \text{ for } \forall\, t \in [0, T]$$

Which shows that $z \in B$ and hence B is convex.

Because of the hypothesis on θ, for all $x \in B$

$||x(t) - x(\bar{t})|| \leq M|t - \bar{t}|$. Thus B is a set of equi-continuous functions. Also B is uniformly bounded, which follows from the fact that each $x \in B$ is uniformly bounded by $g(t)$. And hence by Ascoli's lemma 5.4.7 B is compact.

Q.E.D.

Let U be an open subset of R^m and $Q(t) \subset U$ be a bounded and closed set defined for all $t \in [0, T]$.

Let $Q(t)$ be u.s.c. in t. Let

$$r_2 = \operatorname*{Sup}_t \{ ||u(t)|| , u(t) \in Q(t) \}$$

Consider the DE

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad (5.5.3)$$

where $f: [o, T] \times D \times U \to R^n$. Let $A(t)$ be an $n \times n$ matrix with integrable elements such that

$$\partial[A(t)] \leq -\alpha, \qquad \alpha > o$$

By $\Phi(t, t_o)$ we denote the fundamental matrix of the linear system

$$\dot{x}(t) = A(t) x(t)$$

with $\Phi(t_o, t_o) = I$, $t_o = o$.

Define as before the function F by

$$F(t, x(t), u(t)) = f(t,x(t), u(t)) - A(t) x(t)$$

and assume that F satisfies the property (PP) as mentioned in definition 4.2.1 with f replaced by F and $[o, T] \subset I$. As before the given DE (5.5.3) is equivalent to

$$\dot{x}(t) = A(t) x(t) + F(t,x(t), u(t))$$

which again is equivalent to

$$x(t) = \phi(t,t_o)x_o + \int_{t_o}^{t} \phi(t,s)\ F(s, x(s), u(s))ds$$

for some initial condition $x(t_o) = x_o$.

We are interested in the existence of a measurable function $u(t) \in Q(t)$ and $\delta > o$ such that the system (5.5.3) is g_T stable for any $x_o \in R^n$ with $||x_o|| \leq \delta$.

For any $x \in B$, define the mapping Γx by

$$\Gamma x = \{ y : y(t) = \phi(t,t_o)x_o + \int_{t_o}^{t} \phi(t,s)F(s,x,u)\ ds,\quad u(t) \in Q(t) \} \qquad (5.5.4)$$

Thus Γx defines a set of functions corresponding to a given element x in B, the set being generated by varying $u(t)$ over $Q(t)$.

Suppose that the function $g(t)$ satisfies the relation

$$o < \delta \leq [\ g(t) - \frac{L_2 r_2}{\alpha}\ ]\ e^{\alpha t} - L_1 \int_{o}^{t} e^{\alpha s}\ g(s)\ ds$$

LEMMA 5.5.5

Under the conditions mentioned above $\Gamma x \subset B$ for each $x \in B$.

Proof : Let $y \in \Gamma x$. For convenience let us denote $F(t, x(t), u(t))$ by $F(t)$ only.

$$||y(t)-y(\bar{t})|| \leq ||\phi(t,t_o) - \phi(\bar{t}, t_o)||\delta$$

$$+ ||\int_{t_o}^{t} \phi(t,s)\, F(s)ds - \int_{t_o}^{\bar{t}} \phi(\bar{t},s)F(s)ds||$$

$$\leq ||\phi(t,t_o) - \phi(\bar{t},t_o)||\delta + ||\int_{t_o}^{t} \phi(t,s)F(s)ds$$

$$- \int_{t_o}^{t} \phi(\bar{t},s)F(s)ds|| + ||\int_{t_o}^{t} \phi(t,s)\, F(s)\, ds$$

$$- \int_{t_o}^{t} \phi(t,s)F(s)ds|| + |\int_{t}^{\bar{t}} ||\phi(\bar{t},s)\, F(s)|| ds|$$

$$\leq ||\phi(t,t_o) - \phi(\bar{t},t_o)||\delta + \int_{t_o}^{t} ||\phi(t,s) - \phi(\bar{t},s)||\,.$$

$$||F(s)||ds + |\int_{t}^{\bar{t}} ||\phi(t, s)||\,.||F(s)||\, ds|$$

Since elements of $\Phi(t,s)$ are continuous functions of t for each fixed $s \in [o,T]$, it is uniformly continuous and hence there exists a constant N such that

$$||\phi(t,s) - \phi(\bar{t},s)|| \leq N|t - \bar{t}|$$

for all $s \in [o, T]$. Also

$$\sup_{\substack{s \\ s\leq t}} ||\phi(t,s)|| = \sup_{\substack{s \\ s\leq t}} e^{-\alpha(t-s)} = 1$$

Hence we have

$$||y(t) - y(\bar{t})|| \leq N|t-\bar{t}|\ \delta + \int_{t_o}^{t} N|t-\bar{t}|\,.(L_1r_1+L_2r_2)ds$$

$$+ (L_1r_1+ L_2r_2)|t - \bar{t}|$$

$$\leq [\ N\ \delta + NT(L_1r_1+ L_2r_2) + (L_1r_1+L_2r_2)\ ]|t - \bar{t}|$$

$$= M|t - \bar{t}|$$

where $M = N\delta + (L_1r_1 + L_2r_2).\ (1+NT)$

The above implies that $y \in \theta$. Also

$$||y(t)|| \leq ||\ \phi(t,t_o)||\delta + \int_{t_o}^{t} ||\ \phi(t,s)||\,.||F(s,x(s),u(s)||ds$$

$$\leq e^{-\alpha t}\ \delta + \int_{t_o}^{t} e^{-\alpha(t-s)}(L_1||x(s)||+ L_2||u(s)||)ds$$

$$\leq e^{-\alpha t}\delta + e^{-\alpha t}\int_{0}^{t} e^{\alpha s}\ L_1\ g(s)ds + \frac{L_2r_2}{\alpha}$$

Substituting the value of δ, we get

$$||y(t)\ || \leq\ g(t)$$

Hence $y \in B$ that is $\Gamma\ x \subset B$ **Q.E.D.**

LEMMA 5.5.6

The mapping Γ defined by (5.5.4) is upper semi-continuous.

<u>Proof:</u> From the proof of Lemma 5.5.5 it is clear that the set Γx satisfies the conditions of Ascoli's Lemma, and hence is compact for each $x \in B$. Also because of the continuity of $F(t,x,u)$ with respect to u, the inverse image of any set in Γx is open in $Q(t)$, therefore the upper inverse is also open. Thus by theorem 5.4.4 Γ is u.s.c. <u>Q.E.D.</u>

<u>THEOREM</u> 5.5.7

Let the following conditions be satisfied.

(i) There exists an $n \times n$ matrix $A(t)$ with integrable elements such that $\partial[\, A(t)\,] \leq -\alpha$, $\alpha > 0$.

(ii) The function F defined by

$$F(t,x(t),u(t)) = f(t,x(t),\, u(t)) - A(t)\; x(t)$$

satisfies the property (PP) as mentioned in definition 4.2.1. with f replaced by F and $[\, 0,T\,] \subset I$.

(iii) There exist $Q(t) \subset U \subset R^m$, a closed bounded, u.s.c. set in t with

$$\sup_t \{\, ||u(t)|| \,,\;\; u(t) \in Q(t)\} \;=\; r_2 \,,$$

a number $\delta > 0$ and a bounded non-negative continuous function $g(t)$ such that for all $t \in [\, 0,T\,]$

$$0 < \delta \leq [\, g(t) - \frac{L_2 r_2}{\alpha}\,]\; e^{\alpha t} - L_1 \int_0^t e^{\alpha s} g(s)ds \;.$$

Then for any given $x_o \in S_{r_1}$ with $||x_o|| \leq \delta$, there exists a measurable function $u^*(t) \in Q(t)$ such that the system is g_T stable.

<u>Proof:</u> Suppose that the conditions of the theorem hold. Define the sets B and Γ as in (5.5.1) and (5.5.4) . We show that B and Γ satisfy the theorem 5.4.9.

By lemma 5.5.6 Γ is u.s.c. and hence by theorem 5.4.6, it is closed i.e. Γ satisfies the first condition of theorem 5.4.9. Since for all $x \in B$, $\Gamma x \subset B$, as has been shown in Lemma 5.5.5, $\bigcup \Gamma x$ for all x is also contained in B which is compact. Thus Γ satisfies theorem 5.4.9 and hence there exists an $x^* \in B$ such that

$$x^* \in \Gamma x^*$$

Since $Q(t)$ is closed, bounded and u.s.c. and Γx is its image, by Fillippov's Lemma 5.4.8 corresponding to $x^* \in \Gamma x^*$ there exists a measurable function $u^*(t) \in Q(t)$ such that $x^* = \Gamma(x^*, u^*)$ that is

$$x^* = \Phi(t,t_o)\, x_o + \int_{t_o}^{t} \Phi(t,s) F(s, x^*(s), u^*(s))\, ds$$

And since $x^* \in B \Rightarrow ||x^*(t)|| \leq g(t)$ for all $t \in [\,o, T\,]$, the system 5.2.1 is g_T stable. <u>Q.E.D.</u>

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